Set Membership approximations of predictive control laws: the tradeoff between accuracy and complexity

Technical Report DAUIN_TR_CaFaMiNo_09062009
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I. Problem settings

A. Nonlinear Model Predictive Control

Consider the following nonlinear state space model:

$$x_{t+1} = f(x_t, u_t) \tag{1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system state and control input respectively. In this report, it is assumed that function f in (1) is continuous over $\mathbb{R}^n \times \mathbb{R}^m$. Assume that the control objective is to regulate the system state to the origin under some input and state constraints represented by a compact set $\mathbb{U} \subseteq \mathbb{R}^m$ and a convex set $\mathbb{X} \subseteq \mathbb{R}^n$ respectively, both containing the origin in their interiors. By denoting with N_p and $N_c \leq N_p$ the prediction horizon and the control horizon respectively, the following cost function J can be defined:

$$J(U, x_{t|t}) \doteq \Phi(x_{t+N_p|t}) + \textstyle\sum_{j=0}^{N_p-1} L(x_{t+j|t}, u_{t+j|t})$$

where $x_{t+j|t}$ denotes j step ahead state prediction using the model (1), given the input sequence $u_{t|t}, \ldots, u_{t+j-1|t}$ and the "initial" state $x_{t|t} = x_t$. The final cost Φ and the stage cost L are usually chosen as quadratic functions, tuned according to the desired control performance. $U \doteq [u_{t|t}^T, \ldots, u_{t+N_c-1|t}^T]^T$ is the vector of the control moves to be optimized (where T stands for vector transpose operation). The remaining predicted control moves $[u_{t+N_c|t}^T, \ldots, u_{t+N_p-1|t}^T]^T$ can be computed with different strategies, e.g. by setting $u_{t+j|t} = u_{N_c-1|t}$ or $u_{t+j|t} = K\, x_{t+j|t}$, $\forall j \in [N_c, N_p - 1]$, where K is a suitable matrix. The NMPC control law is then obtained applying the following RH strategy:

1) At time instant t, get x_t.
2) Solve the optimization problem:

$$\min_{U} \quad J(U, x_{t|t}, N_p) \tag{2}$$

s. t.

$$x_{t+j|t} \in \mathbb{X},\ j = 1, \ldots, N_p \tag{3}$$

$$u_{t+j|t} \in \mathbb{U},\ j = 0, \ldots, N_p \tag{4}$$

3) Apply the first element of the solution sequence U to the optimization problem as the actual control action $u_t = u_{t|t}$.
4) Repeat from step 1) at time $t + 1$.

Additional constraints (e.g. state contraction, terminal set, etc.…) may be also employed in order to achieve stability of the closed–loop system. It is assumed that the optimization problem (2) is feasible over a set $\mathcal{F} \subseteq \mathbb{R}^n$ which will be referred to as the "feasibility set". The application of such RH procedure implicitly defines the predictive controller as a nonlinear static function κ^0 of the state, i.e.:

$$u_t = \kappa^0(x_t)$$

In this report, it is supposed that the nominal control law κ^0 asymptotically stabilizes the origin of the closed–loop system and that it is continuous over the feasibility set $\mathcal{F}$. Such property depends on the characteristics of the optimization problem (2), see at this regard e.g. the recent work of [1] and the references therein.

B. Stabilizing properties of approximated NMPC laws

In the standard NMPC formulation, the nominal control law κ^0 is evaluated by solving the optimization problem (2) on–line. However, a limitation in the practical use of NMPC is the presence of fast plant dynamics, for which the required sampling time may be too low for real-time optimization. A viable solution to this problem is the use of an approximated control function $\hat{\kappa} \approx \kappa^0$, derived off–line, whose on–line evaluation is more efficient. It is considered that $\hat{\kappa}$ is defined over a compact set $\mathcal{X}$, containing the origin in its interior:

$$\hat{\kappa} : \mathcal{X} \to \mathbb{R},\ \mathcal{X} \subseteq \mathcal{F}$$

Moreover, $\hat{\kappa}$ is computed on the basis of the knowledge of a finite number ν of exact control moves, i.e.:

$$\tilde{u}^k = \kappa^0(\tilde{x}^k), k = 1, \ldots, \nu \tag{5}$$

where the state values $\tilde{x}^k$ are suitably chosen and define the set:

$$\mathcal{X}_\nu = \{\tilde{x}^k,\ k = 1, \ldots, \nu\} \subseteq \mathcal{F}$$

Note that the points $\tilde{x}^k$, $k = 1, \ldots, \nu$ must be inside the feasibility set $\mathcal{F}$ (in order to be able to compute the corresponding exact MPC control move $\tilde{u}^k$) but need not to belong to the set $\mathcal{X}$. A major issue arising in the use of approximated NMPC is about the guaranteed closed–loop performance obtained when $\hat{\kappa}$ is employed instead of κ^0. To this regard, it has been proved (see e.g. [2]–[3]) that if $\hat{\kappa}$ has the following *key properties*:

i) Input constraint satisfaction:

$$\hat{\kappa}(x) \in \mathbb{U}\ \forall x \in \mathcal{X} \tag{6}$$

This work has been partly supported by Regione Piemonte, Italy, under the Project "Power kites for naval propulsion".

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ii) The pointwise approximation error $\Delta_{\hat{\kappa}}(x) = \kappa^0(x) - \hat{\kappa}(x)$ is bounded:

$$\|\Delta_{\hat{\kappa}}(x)\| \leq \zeta, \ \forall x \in \mathcal{X} \tag{7}$$

where $\|\cdot\|$ is a suitable norm, e.g. Euclidean.

iii) The bound $\zeta(\nu)$ converges to zero:

$$\lim_{\nu \to \infty} \zeta(\nu) = 0 \tag{8}$$

then it is always possible to explicitly compute a suitable finite value of ν, such that there exist a finite value $\Delta \in \mathbb{R}^+$ with the following characteristics (see [3] for details):

i) the distance between the closed loop state trajectories obtained with the nominal and the approximated control laws is bounded by $\Delta(\nu)$, which can be explicitly computed

ii) $\lim_{\nu \to \infty} \Delta(\nu) = 0$

iii) the closed loop state trajectories, obtained when the approximated control law is used, are kept inside the compact set $\mathcal{X}$, and asymptotically converge to an arbitrarily small neighborhood of the origin

As regards the asymptotic behaviour of ζ as $\nu \to \infty$, it is assumed that $\mathcal{X}_\nu$ is chosen such that the following property holds:

$$\lim_{\nu \to \infty} d_H(\mathcal{X}, \mathcal{X}_\nu) = 0 \tag{9}$$

where $d_H(X, X_\nu)$ is the Hausdorff distance between $\mathcal{X}$ and $\mathcal{X}_\nu$ (see e.g. [4]):

$$d_H(\mathcal{X}, \mathcal{X}_\nu) \doteq \max \left(\sup_{x \in \mathcal{X}} \inf_{\tilde{x} \in \mathcal{X}_\nu} (\|x - \tilde{x}\|_2), \sup_{\tilde{x} \in \mathcal{X}_\nu} \inf_{x \in \mathcal{X}} (\|x - \tilde{x}\|_2) \right)$$

Note that uniform gridding over $\mathcal{X}$ satisfies condition (9).

Properties (6)–(8) imply that there exists a finite and computable bound on the worst–case accuracy obtained with the employed approximated control law, and that such a bound decreases to zero as ν grows. However, larger values of ν usually correspond to higher memory usage and computational times (both on–line and off–line), leading to the need of choosing a tradeoff between accuracy, computational efficiency and employed memory. In [2]–[3], two different SM techniques have been proposed, both satisfying properties (6)–(8), whose performance are complementary. In particular, the optimal (OPT) SM approximation [2] gives the minimal worst–case error, but higher computational complexity with respect to the nearest point (NP) approach [5], whose guaranteed accuracy is worse but with much more efficient computation. The aim of this report is to provide new approaches to approximate an NMPC law, able to achieve better tradeoffs between accuracy and complexity. The properties of the OPT and NP techniques are now briefly recalled, as they are instrumental to introduce the main results presented in this work.

II. Set membership approximations of NMPC with guaranteed accuracy

A. Prior information

As already pointed out, the approximation of function κ^0 is performed on a compact set $\mathcal{X} \subseteq \mathcal{F}$. Since $\mathcal{X}$ and the image set $\mathbb{U}$ of κ^0 are compact, continuity of κ^0 over $\mathcal{F}$ implies that its components κ_i^0, $i = 1, \ldots, m$ are Lipschitz continuous functions over $\mathcal{X}$, i.e. there exist finite constants γ_i, $i = 1, \ldots, m$ such that $\forall x^1, x^2 \in \mathcal{X}$, $\forall i \in [1, m], |\kappa_i^0(x^1) - \kappa_i^0(x^2)| \leq \gamma_i \|x^1 - x^2\|_2$, and $\forall x^1, x^2 \in \mathcal{X}, \|\kappa^0(x^1) - \kappa^0(x^2)\|_2 \leq \|\gamma\|_2 \|x^1 - x^2\|_2$, where $\gamma = [\gamma_1, \ldots, \gamma_m]^T$. This prior information, together with the knowledge of the off–line computed values (5), can be summarized by concluding that

$$\kappa^0 \in FFS, \tag{10}$$

where the set FFS (Feasible Functions Set) is defined as $FFS = \{\kappa : \mathcal{X} \to \mathbb{U}, \kappa = [\kappa_1, \ldots, \kappa_m]^T : \kappa_i \in FFS_i, \forall i \in [1, m]\}$, with $FFS_i = \{\kappa_i \in \mathcal{A}_{\gamma_i} : \kappa_i(\tilde{x}^k) = \tilde{u}_i^k, \ k = 1, \ldots, \nu\}$ and $\mathcal{A}_{\gamma_i}$ being the set of all continuous Lipschitz functions κ_i on $\mathcal{X}$, with constant γ_i. As regards the computation of the Lipschitz constants $\gamma = [\gamma_1, \ldots \gamma_m]$, which are needed to apply the presented SM techniques, estimates $\hat{\gamma}_i, i = 1, \ldots, m$ can be derived off–line as follows:

$$\hat{\gamma}_i = \inf \left(\tilde{\gamma}_i : \tilde{u}_i^h + \tilde{\gamma}_i \|\tilde{x}^h - \tilde{x}^k\|_2 \geq \tilde{u}_i^k, \ \forall k, h = 1, \ldots, \nu \right) \tag{11}$$

Convergence of $\hat{\gamma}_i$ to $\gamma_i, i = 1, \ldots, m$ has been showed e.g. in [3], i.e. $\lim_{\nu \to \infty} \hat{\gamma}_i = \gamma_i, \ \forall i = 1, \ldots, m$.

B. Optimal SM approximation

The OPT approximation technique and its accuracy and stabilizing properties have been described in [2] and [3], for the case of linear systems. With the OPT approach, for a given value of ν the obtained approximating function κ^{OPT} gives the minimal worst–case approximation error.

For each function κ_i^0, $i \in [1, m]$ and for a given value $x \in \mathcal{X}$, an approximation $\kappa_i^{\text{OPT}}(x) \approx \kappa_i^0(x)$ is computed as (see [3]):

$$\kappa_i^{\text{OPT}}(x) = \tfrac{1}{2}[\overline{\kappa}_i(x) + \underline{\kappa}_i(x)] \in FFS_i \tag{12}$$

where $\overline{\kappa}_i(x) = \sup_{\tilde{\kappa}_i \in FFS_i} \tilde{\kappa}_i(x)$ and $\underline{\kappa}_i(x) = \inf_{\tilde{\kappa}_i \in FFS_i} \tilde{\kappa}_i(x)$, called "optimal bounds", are the tightest upper and lower bounds of $\kappa_i^0(x)$, according to the considered prior information. As shown in [3], the optimal bounds can be computed as:

$$\begin{aligned} \overline{\kappa}_i(x) &\doteq \min \left[\overline{u}_i, \min_{k=1,\ldots,\nu} \left(\tilde{u}_i^k + \gamma_i \|x - \tilde{x}^k\|_2 \right) \right] \in FFS_i \\ \underline{\kappa}_i(x) &\doteq \max \left[\underline{u}_i, \max_{k=1,\ldots,\nu} \left(\tilde{u}_i^k - \gamma_i \|x - \tilde{x}^k\|_2 \right) \right] \in FFS_i \end{aligned} \tag{13}$$

Function κ_i^{OPT} (12) is such that the related *guaranteed approximation error* $E(\kappa_i^{\text{OPT}}) \doteq \sup_{\tilde{\kappa}_i \in FSS_i} \|\tilde{\kappa}_i - \kappa_i^{\text{OPT}}\|_p$ is minimal for any L_p–norm, with $p \in [1, \infty]$, and it is therefore equal to the *radius of information* $r_{p,i}$ (see e.g. [6]).

Then, the optimal SM approximation $\kappa^{\text{OPT}} \approx \kappa^0$ is defined as:

$$\kappa^{\text{OPT}} = \left[\kappa_1^{\text{OPT}}, \ldots, \kappa_m^{\text{OPT}} \right]^T \tag{14}$$

It can be proved that ([3]):

$$\kappa^{\text{OPT}} : \mathcal{X} \to \mathbb{U} \tag{15}$$

$$\begin{aligned} &\|\Delta_{\kappa^{\text{OPT}}}(x)\|_2 \doteq \|\kappa^0(x) - \kappa^{\text{OPT}}(x)\|_2 \leq \\ &\leq e^{\text{OPT}}(x) \leq \|r_\infty\|_2 = \zeta^{\text{OPT}}, \ \forall x \in \mathcal{X} \end{aligned} \tag{16}$$

$$\lim_{\nu \to \infty} \zeta^{\text{OPT}}(\nu) = 0 \tag{17}$$

where $r_\infty \doteq [r_{\infty,1}, \dots, r_{\infty,m}]$ and $e^{\text{OPT}}(x) \doteq \|\frac{1}{2}(\overline{\kappa}_i(x) - \underline{\kappa}_i(x))\|_2,\ i \in [1,m]$. Thus, function κ^{OPT} satisfies the key properties (6)–(8). As regards the computation of $r_{\infty,i}, i = 1,\dots,m$, the results presented in [7] can be employed.

C. Nearest point SM approximation

The NP approach, originally introduced in [5], gives an approximating function whose accuracy is not the optimal one, but whose computation is simpler. In particular, for a given value of ν, the NP approximation leads in general to a higher approximation error bound ζ^{NP} than OPT approximation, but to lower on–line computation times, whose growth as a function of ν is much slower than that of OPT approximation. Thus, the NP approximation required to guarantee given stability and performance properties may need much lower on–line computation times with respect to OPT approximation, at the expenses of longer off–line computation time and higher memory usage.
For any $x \in \mathcal{X}$, denote with $\tilde{x}^{\text{NP}} \in \mathcal{X}_\nu$ a state value such that $\|\tilde{x}^{\text{NP}} - x\|_2 = \min_{\tilde{x}\in\mathcal{X}_\nu} \|\tilde{x} - x\|_2$. Then, the NP approximation $\kappa^{\text{NP}}(x)$ is computed as:

$$\kappa^{\text{NP}}(x) \doteq \kappa^0(\tilde{x}^{\text{NP}}) \tag{18}$$

Such approximation trivially satisfies condition (6). Moreover, it can be shown that also the key properties (7), (8) are satisfied (see [5]):

$$\begin{aligned}&\|\Delta_{\kappa^{\text{NP}}}(x)\|_2 = \|\kappa^0(x) - \kappa^{\text{NP}}(x)\|_2 \le \|\gamma\|_2 \|x - \tilde{x}^{\text{NP}}\|_2 = \\ &= e^{\text{NP}}(x) \le \zeta^{\text{NP}} = \|\gamma\|_2\, d_H(\mathcal{X}, \mathcal{X}_\nu),\ \forall x \in \mathcal{X}\end{aligned} \tag{19}$$

$$\lim_{\nu\to\infty} \zeta^{\text{NP}}(\nu) = 0 \tag{20}$$

It can be proved that, for a given value of ν, $\zeta^{\text{OPT}} \le \zeta^{\text{NP}}$, thus the same guaranteed accuracy can be obtained with a lower ν value by OPT approach with respect to NP. However, the evaluation efficiency of NP can be much higher than that of OPT.
In this report two alternative techniques are described, giving different compromises between accuracy, memory usage and evaluation complexity with respect to OPT and NP. The accuracy properties of these new approaches will be investigated in the SM framework. Since both the described techniques give a non–minimal worst–case accuracy (as it also happens with the NP approach), they will be referred to as "suboptimal" SM approximations. For simplicity of notation, in the remaining of the report it will be assumed that $\kappa^0 : \mathbb{R}^n \to \mathbb{R}$ and that $\mathbb{U} = \{u \in \mathbb{R} : \underline{u} \le u \le \overline{u}\}$, where $\underline{u}, \overline{u} \in \mathbb{R}$.

III. Suboptimal approximations of NMPC

In this section two suboptimal approximation methods are presented, with improved evaluation efficiency with respect to the OPT technique and improved accuracy with respect to the NP approach, and their accuracy properties are investigated in the framework of SM approximation.

A. Piecewise linear approximation

Let $X_1, X_2, ..., X_q$ be a triangulation defined by the set of points $\mathcal{X}_\nu$. Such a triangulation is a collection of sets $X_1, X_2, ..., X_q$ such that

$$\begin{aligned}&\bigcup_{j=1}^{q} X_j = \text{chull}(\mathcal{X}_\nu),\\ &\text{int}(X_h) \cap \text{int}(X_j) = 0 \text{ for } h \ne j,\\ &\text{all } X_j\text{'s are simplices (triangles for } n = 2),\\ &\text{the vertices of the simplices are points of } \mathcal{X}_\nu,\\ &\text{all points of } \mathcal{X}_\nu \text{ are vertices of the simplices.}\end{aligned}$$

Here $\text{int}(\cdot)$ denotes the interior of a set and $\text{chull}(\cdot)$ denotes the convex hull of a set. A triangulation partitions the convex hull of $\mathcal{X}_\nu$ into a set of simplices, which will be also referred to as "triangles" in the following. For each triangle X_j, consider the set of points

$$\{\tilde{x}^{j,k},\, k = 1,\dots,n+1 :\ \tilde{x}^{j,k} \text{ is a vertex of } X_j\}$$

Since a triangle has $n+1$ vertices, such a set contains $n+1$ points in $\mathbb{R}^n$. Let $K_j x + Q_j$ be the hyperplane interpolating these points. The coefficients $K_j \in \mathbb{R}^n$, $Q_j \in \mathbb{R}$ can be trivially obtained as

$$\begin{bmatrix} K_j^T \\ Q_j \end{bmatrix} = \begin{bmatrix} \tilde{x}^{j,1\,T} & 1 \\ \dots & \dots \\ \tilde{x}^{j,n+1\,T} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{u}^{j,1} \\ \dots \\ \tilde{u}^{j,n+1} \end{bmatrix} \tag{21}$$

where $\tilde{u}^{j,k} = \kappa_i^0(\tilde{x}^{j,k})$. Assume that $\mathcal{X} \subseteq \text{chull}(\mathcal{X}_\nu)$ and define the piecewise linear approximation (LIN)

$$\kappa^{\text{LIN}}(x) \doteq K_{\hat{j}} x + Q_{\hat{j}} \tag{22}$$

where $\hat{j} \in \arg\min_{j=1,\dots,q} d(x, X_j)$ and $d(x, X) \doteq \inf_{\xi\in X}(\|x - \xi\|_2)$ is the distance between the point x and the set X. Clearly, for given $x \in \mathcal{X}$, $X_{\hat{j}}$ is a triangle which contains x. If $x \in \text{int}(X_{\hat{j}})$, this triangle is unique. According to the above definition, κ^{LIN} is a continuous piecewise linear function, which can be used to approximate κ^0. The next result shows that that κ^{LIN} satisfies the properties (6)–(8). Define the approximation error:

$$\Delta_{\kappa^{\text{LIN}}}(x) \doteq \kappa^0(x) - \kappa^{\text{LIN}}(x), \tag{23}$$

Theorem 1: The following properties hold:

i) $\kappa^{\text{LIN}}(x) \in \mathbb{U},\ \forall x \in \mathcal{X}$.

ii) The pointwise approximation error $\Delta_{\kappa^{\text{LIN}}}(x)$ of κ^{LIN} is bounded as

$$\begin{aligned}&\forall x \in \mathcal{X},\ |\Delta_{\kappa^{\text{LIN}}}(x)| \le e^{\text{LIN}}(x) = \\ &= |\kappa^{\text{OPT}}(x) - \kappa^{\text{LIN}}(x)| + e^{\text{OPT}}(x) \ge e^{\text{OPT}}(x)\\ &\forall x \in \mathcal{X},\ e^{\text{LIN}}(x) \le \zeta^{\text{LIN}} = \sup_{x\in\mathcal{X}} e^{\text{LIN}}(x) \ge \zeta^{\text{OPT}}\end{aligned}$$

iii) $\lim_{\nu\to\infty} \zeta^{\text{LIN}}(\nu) = 0$.

Proof:

i) For any $x \in \mathcal{X}$, consider the the vertices $\tilde{x}^l, l = 1,\dots,n+1$ of the partition $X_{\hat{j}} : \hat{j} \in \arg\min_{j=1,\dots,q} d(x, X_j)$, and the corresponding exact control moves $\tilde{u}^l = \kappa^0(\tilde{x}^l)$. Note that $\tilde{u}^l =$

$\kappa^{\text{LIN}}(\tilde{x}^l)$ by definition (21). The point x can be expressed as:

$$x = \sum_{l=1}^{n+1} w_l \, \tilde{x}^l, \; w_l > 0 \, \forall l \in [1, n+1], \; \sum_{l=1}^{n+1} w_l = 1$$

and the approximated control move $\kappa^{\text{LIN}}(x)$ can be therefore computed as:

$$\kappa^{\text{LIN}}(x) = \sum_{l=1}^{n+1} w_l \, (K_j \tilde{x}^l + Q_j) = \\ = \sum_{l=1}^{n+1} w_l \, \kappa^{\text{LIN}}(\tilde{x}^l) = \sum_{l=1}^{n+1} w_l \, \tilde{u}^l$$

thus it can be noted that:

$$\kappa^{\text{LIN}}(x) \leq \max_{l=1,\dots,n+1} (\tilde{u}^l) \sum_{l=1}^{n+1} w_l = \max_{l=1,\dots,n+1} (\tilde{u}^l) \leq \overline{u}$$
$$\kappa^{\text{LIN}}(x) \geq \min_{l=1,\dots,n+1} (\tilde{u}^l) \sum_{l=1}^{n+1} w_l = \min_{l=1,\dots,n+1} (\tilde{u}^l) \geq \underline{u}$$
$$\Rightarrow \kappa^{\text{LIN}}(x) \in \mathbb{U}$$

ii) $|\Delta_{\kappa^{\text{LIN}}}(x)| = |\kappa^0(x) - \kappa^{\text{LIN}}(x)| = |\kappa^{\text{OPT}}(x) + \Delta_{\kappa^{\text{OPT}}}(x) - \kappa^{\text{LIN}}(x)| \leq |\kappa^{\text{OPT}}(x) - \kappa^{\text{LIN}}(x)| + |\Delta_{\kappa^{\text{OPT}}}(x)| \leq |\kappa^{\text{OPT}}(x) - \kappa^{\text{LIN}}(x)| + e^{\text{OPT}}(x) = e^{\text{LIN}}(x) \geq e^{\text{OPT}}(x)$

$$e^{\text{LIN}}(x) \leq \sup_{x \in \mathcal{X}} e^{\text{LIN}}(x) \doteq \zeta^{\text{LIN}} \geq \sup_{x \in \mathcal{X}} e^{\text{OPT}}(x) = \zeta^{\text{OPT}}$$

iii) Considering that $\lim_{\nu \to \infty} e^{\text{OPT}}(x) = 0, \, \forall x \in \mathcal{X}$ (i.e. $\lim_{\nu \to \infty} \kappa^{\text{OPT}}(x) = \kappa^0(x)$, see [2], [3]) and that, since κ^0 is Lipschitz continuous, $\lim_{\nu \to \infty} \kappa^{\text{LIN}}(x) = \kappa^0(x), \, \forall x \in \mathcal{X}$, it can be noted that:

$$\forall x \in \mathcal{X}, \; \lim_{\nu \to \infty} e^{\text{LIN}}(x) = \\ = \lim_{\nu \to \infty} |\kappa^{\text{OPT}}(x) - \kappa^{\text{LIN}}(x)| + \lim_{\nu \to \infty} e^{\text{OPT}}(x) = \\ |\kappa^0(x) - \kappa^0(x)| + 0 = 0$$

thus

$$\lim_{\nu \to \infty} \zeta^{\text{LIN}}(\nu) = \lim_{\nu \to \infty} \sup_{x \in \mathcal{X}} e^{\text{LIN}}(x, \nu) = 0$$

■

B. SM Neighborhood approximation

Let $X_1, X_2, ..., X_q$ be a collection of sets such that

$$\mathcal{X} \subseteq \bigcup_{j=1}^{q} X_j.$$

For any $x \in \mathcal{X}$, let $\hat{\jmath} \in \arg \min_{j=1,\dots,q} d(x, X_j)$, so that $X_{\hat{\jmath}}$ contains x. Define the sets of indices

$$P_j \doteq \left\{ k : \tilde{x}^k \in X_j \cup \{\tilde{x}^{\text{NP}}\} \right\}, \; j = 1, \dots, q \tag{24}$$

The SM neighborhood (NB) approximation of κ^0 is given by:

$$\kappa^{\text{NB}}(x) \doteq \frac{1}{2} [\overline{\kappa}^{\text{NB}}(x) + \underline{\kappa}^{\text{NB}}(x)] \tag{25}$$

with

$$\overline{\kappa}^{\text{NB}}(x) \doteq \min \left[\overline{u}, \min_{k \in P_{\hat{\jmath}}} \left(\tilde{u}^k + \gamma \|x - \tilde{x}^k\|_2 \right) \right] \\ \underline{\kappa}^{\text{NB}}(x) \doteq \max \left[\underline{u}, \max_{k \in P_{\hat{\jmath}}} \left(\tilde{u}^k - \gamma \|x - \tilde{x}^k\|_2 \right) \right]. \tag{26}$$

Note that the function κ^{NB} is defined similarly to κ^{OPT}, except that only a subset of points of $\mathcal{X}_\nu$ is used to compute $\overline{\kappa}^{\text{NB}}(x)$ and $\underline{\kappa}^{\text{NB}}(x)$. In order to show that κ^{NB} satisfies the properties (6)–(8) and to evaluate its accuracy, let us define the indices

$$\overline{k} \doteq \arg \min_{k=1,\dots,\nu} (\tilde{u}^k + \gamma \|x - \tilde{x}^k\|_2) \\ \underline{k} \doteq \arg \max_{k=1,\dots,\nu} (\tilde{u}^k - \gamma \|x - \tilde{x}^k\|_2) \\ \overline{\jmath} \doteq \arg \min_{k \in P_{\hat{\jmath}}} \left(\tilde{u}^k + \gamma \|x - \tilde{x}^k\|_2 \right) \\ \underline{\jmath} \doteq \arg \max_{k \in P_{\hat{\jmath}}} \left(\tilde{u}^k - \gamma \|x - \tilde{x}^k\|_2 \right)$$

Moreover, define the following scalar quantities:

$$\delta(x) = \gamma(\|\tilde{x}^{\overline{k}} - \tilde{x}^{\overline{\jmath}}\|_2 + \|\tilde{x}^{\underline{k}} - \tilde{x}^{\underline{\jmath}}\|_2) \tag{27}$$
$$\Delta_{\kappa^{\text{NB}}}(x) \doteq \kappa^0(x) - \kappa^{\text{NB}}(x)$$

Theorem 2: The following properties hold:

i) $\kappa^{\text{NB}}(x) \in \mathbb{U}, \, \forall x \in \mathcal{X}$.

ii) The pointwise approximation error $\Delta_{\kappa^{\text{NB}}}(x)$ of κ^{NB} is bounded as

$$\forall x \in \mathcal{X}, \, |\Delta_{\kappa^{\text{NB}}}(x)| \leq \\ \leq e^{\text{NB}}(x) \doteq \min(e^{\text{NP}}, e^{\text{OPT}}(x) + \delta(x)) \\ \forall x \in \mathcal{X}, \, e^{\text{NB}}(x) \leq \zeta^{\text{NB}} = \sup_{x \in \mathcal{X}} e^{\text{NB}} \leq \zeta^{\text{NP}}$$

iii) The following convergence property holds:

$$\lim_{\nu \to \infty} \zeta^{\text{NB}}(\nu) = 0.$$

iv) If $\overline{k} = \overline{\jmath}$ and $\underline{k} = \underline{\jmath}$ then

$$\kappa^{\text{NB}}(x) = \kappa^{\text{OPT}}(x).$$

Proof:

i) From (25)–(26) it can be noted that, for any $x \in \mathcal{X}$:

$$\kappa^{\text{NB}}(x) = \frac{1}{2}(\overline{\kappa}^{\text{NB}} + \underline{\kappa}^{\text{NB}}) \leq \overline{\kappa}^{\text{NB}} \leq \overline{u} \\ \kappa^{\text{NB}}(x) = \frac{1}{2}(\overline{\kappa}^{\text{NB}} + \underline{\kappa}^{\text{NB}}) \geq \underline{\kappa}^{\text{NB}} \geq \underline{u} \\ \Rightarrow \kappa^{\text{NB}}(x) \in \mathbb{U}$$

ii) For any $x \in \mathcal{X}$, note that (from (13) and (26)):

$$\underline{\kappa}^{\text{NB}}(x) \leq \underline{\kappa}^{\text{OPT}}(x) \leq \kappa^0(x) \leq \overline{\kappa}^{\text{OPT}}(x) \leq \overline{\kappa}^{\text{NB}}(x) \\ \underline{\kappa}^{\text{NB}}(x) - \kappa^{\text{NB}}(x) \leq \kappa^0(x) - \kappa^{\text{NB}}(x) \leq \overline{\kappa}^{\text{NB}}(x) - \kappa^{\text{NB}}(x) \\ -\frac{1}{2}(\overline{\kappa}^{\text{NB}}(x) - \underline{\kappa}^{\text{NB}}(x)) \leq \kappa^0(x) - \kappa^{\text{NB}}(x) \leq \\ \leq \frac{1}{2}(\overline{\kappa}^{\text{NB}}(x) - \underline{\kappa}^{\text{NB}}(x))$$

$$\Rightarrow |\kappa^0(x) - \kappa^{\text{NB}}(x)| \leq \frac{1}{2}(\overline{\kappa}^{\text{NB}}(x) - \underline{\kappa}^{\text{NB}}(x)) \tag{28}$$

Consider now the distance between the optimal upper bound $\overline{\kappa}^{\text{OPT}}(x)$ (13) and the suboptimal upper bound $\overline{\kappa}^{\text{NB}}(x)$ (26). Since by definition $\overline{\kappa}^{\text{OPT}}(x) \leq$

$\overline{\kappa}^{\text{NB}}(x) \leq \overline{u}$, if $\overline{\kappa}^{\text{OPT}}(x) = \overline{u}$ then $\overline{\kappa}^{\text{NB}}(x) - \overline{\kappa}^{\text{OPT}}(x) = \overline{u} - \overline{u} = 0$. Otherwise note that:

$$\begin{aligned}
&0 < \overline{\kappa}^{\text{NB}}(x) - \overline{\kappa}^{\text{OPT}}(x) \leq \\
&\tilde{u}^{\overline{j}} + \gamma\|x - \tilde{x}^{\overline{j}}\|_2 - \tilde{u}^{\overline{k}} - \gamma\|x - \tilde{x}^{\overline{k}}\|_2 \leq \\
&\leq \gamma\|\tilde{x}^{\overline{j}} - \tilde{x}^{\overline{k}}\|_2 + \gamma\|x - \tilde{x}^{\overline{j}} - x + \tilde{x}^{\overline{k}}\|_2 = \\
&= 2\gamma\|\tilde{x}^{\overline{j}} - \tilde{x}^{\overline{k}}\|_2
\end{aligned}$$

Similarly,it can be obtained that:

$$\begin{aligned}
&0 < \underline{\kappa}^{\text{OPT}}(x) - \underline{\kappa}^{\text{NB}}(x) \leq \\
&\tilde{u}^{\underline{k}} - \gamma\|x - \tilde{x}^{\underline{k}}\|_2 - \tilde{u}^{\underline{j}} + \gamma\|x - \tilde{x}^{\underline{j}}\|_2 \leq \\
&\leq \gamma\|\tilde{x}^{\underline{j}} - \tilde{x}^{\underline{k}}\|_2 + \gamma\|x - \tilde{x}^{\underline{j}} - x + \tilde{x}^{\underline{k}}\|_2 = \\
&= 2\gamma\|\tilde{x}^{\underline{j}} - \tilde{x}^{\underline{k}}\|_2
\end{aligned}$$

thus, the distance between the OPT and NB approximations is bounded:

$$\begin{aligned}
&|\kappa^{\text{OPT}}(x) - \kappa^{\text{NB}}(x)| = \\
&= \frac{1}{2}|\overline{\kappa}^{\text{OPT}}(x) + \underline{\kappa}^{\text{OPT}}(x) - \overline{\kappa}^{\text{NB}}(x) - \underline{\kappa}^{\text{NB}}(x)| \leq \\
&\leq \frac{1}{2}(|\overline{\kappa}^{\text{OPT}}(x) - \overline{\kappa}^{\text{NB}}(x)| + |\underline{\kappa}^{\text{OPT}}(x) - \underline{\kappa}^{\text{NB}}(x)|) \leq \\
&\leq \gamma(\|\tilde{x}^{\overline{j}} - \tilde{x}^{\overline{k}}\|_2 + \|\tilde{x}^{\underline{j}} - \tilde{x}^{\underline{k}}\|_2) = \delta(x)
\end{aligned} \tag{29}$$

Consequently, note that:

$$\begin{aligned}
&|\Delta_{\kappa^{\text{NB}}}(x)| = |\kappa^0(x) - \kappa^{\text{NB}}(x)| \leq \\
&\leq |\kappa^0(x) - \kappa^{\text{OPT}}(x) + (\kappa^{\text{OPT}}(x) - \kappa^{\text{NB}}(x))| \leq \\
&\leq e^{\text{OPT}}(x) + \delta(x),\ \forall x \in \mathcal{X}
\end{aligned} \tag{30}$$

At the same time, since by construction (24) for any $x \in \mathcal{X}$ the set of points $\{\tilde{x}^j : j \in P_{\hat{j}}\}$ contains the nearest neighbor $\tilde{x}^{\text{NP}}$ of x, it can be noted that (from (26)):

$$\begin{aligned}
\overline{\kappa}^{\text{NB}} &\leq \tilde{u}^{\text{NP}} + \gamma\|x - \tilde{x}^{\text{NP}}\|_2 \\
\underline{\kappa}^{\text{NB}} &\geq \tilde{u}^{\text{NP}} - \gamma\|x - \tilde{x}^{\text{NP}}\|_2
\end{aligned}$$

Thus, from (28):

$$\begin{aligned}
&|\Delta_{\kappa^{\text{NB}}}(x)| = |\kappa^0(x) - \kappa^{\text{NB}}(x)| \leq \\
&\frac{1}{2}(\overline{\kappa}^{\text{NB}}(x) - \underline{\kappa}^{\text{NB}}(x)) \leq \\
&\leq \frac{1}{2}(\tilde{u}^{\text{NP}} + \gamma\|x - \tilde{x}^{\text{NP}}\|_2 - \tilde{u}^{\text{NP}} + \gamma\|x - \tilde{x}^{\text{NP}}\|_2) = \\
&= \gamma\|x - \tilde{x}^{\text{NP}}\|_2 = e^{\text{NP}}
\end{aligned} \tag{31}$$

By considering the tightest bound between (30) and (31), it can be obtained that:

$$\begin{aligned}
&|\Delta_{\kappa^{\text{NB}}}(x)| \leq \min(e^{\text{NP}}(x), e^{\text{OPT}}(x) + \delta(x)) = e^{\text{NB}}(x) \leq \\
&\leq e^{\text{NP}}(x),\ \forall x \in \mathcal{X} \\
&\zeta^{\text{NB}} = \sup_{x \in \mathcal{X}} e^{\text{NB}} \leq \gamma d_H(\mathcal{X}, \mathcal{X}_\nu) = \zeta^{\text{NP}}
\end{aligned}$$

iii) Trivially follows from (31) and the properties of the NP approximation (20)

iv) Trivially follows from (29) by using $\overline{k} = \overline{j}$ and $\underline{k} = \underline{j}$

■

Remark 1: For given number of data ν, under suitable choices of the sets $X_1, X_2, ..., X_q$ and using efficient search algorithms, both LIN and NB approximations lead to a significantly better on–line computational efficiency than the OPT approximation, at the expense of higher memory usage and some degradation of the worst case approximation error. Note however that, in practical application, such a degradation does not necessarily imply that the performance of the suboptimal techniques are worse than those of the optimal one.

REFERENCES

[1] J. Spjøtvold, P. Tøndel, and T. A. Johansen, "Continuous selection and unique polyhedral representation of solutions to convex parametric quadratic programs," *Journal of Optimization Theory and Applications*, vol. 134, pp. 177–189, 2007.

[2] M. Canale, L. Fagiano, and M. Milanese, "Fast implementation of predictive controllers using sm approximation methodologies," in *46th Conference on Decision and Control*, 2007, pp. 1361–1367.

[3] ——, "Set membership approximation theory for fast implementation of model predictive control laws," *Automatica*, to appear.

[4] C. S. Blagovest, *Hausdorff Approximations*. Springer, 1990.

[5] M. Canale, L. Fagiano, and M. Milanese, "Fast nonlinear model predictive control using set membership approximation," in *17th IFAC World Congress*, Seoul, Korea, 2008, pp. 12 165–12 170.

[6] J. Traub and H. Woźniakowski, *A General Theory of Optimal Algorithms*. New York: Academic Press, 1980.

[7] M. Milanese and C. Novara, "Computation of local radius of information in sm-ibc identification of nonlinear systems," *Journal of Complexity*, vol. 4, no. 6, pp. 937–951, August 2007.