Robust Model Predictive Control via Scenario Optimization
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Proof of Theorem 4.1

G.C. Calafiore*, L. Fagiano*,**

Preliminaries. Notice first that, for any $t \geq 0$, if $x_t \in \mathbb{X}_f$ then the optimal solution to problem $\mathcal{P}(x_t, \omega_t)$ is $(\mathcal{V}_t^*, z_t^*, q_t^*) = (0, 0, 0)$, since the terminal control law (i.e., with $\mathcal{V}_t = 0$) is able to keep the predicted state trajectory in the terminal set while satisfying all constraints. Also, if $z_t^* = 0$ is the optimal objective of problem $\mathcal{P}(x_t, \omega_t)$, then $x_t \in \mathbb{X}_f$, since z_t^* is an upper bound of $d(x_t, \mathbb{X}_f)$ (see also Remark 3.1), therefore $z_t^* = 0 \iff x_t \in \mathbb{X}_f$. Let then $x_t \notin \mathbb{X}_f$.

Proof of statement (a). At time $t = 0$, Proposition 3.1 guarantees with practical certainty that the first control correction satisfies the constraints on u_0 and x_1 with probability no less than p and constraint violation $q_t = q_t^*$. At any generic time step $t \geq 1$, the variables $(\tilde{\mathcal{V}}_t, \tilde{z}_t, \tilde{q}_t)$ are computed. Then, two cases may occur. If $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then case 3.c) is detected, and the first element $v_{0|t}^*$ of the optimal sequence $\mathcal{V}_t^*$ is applied to the system. Being this sequence the solution of a scenario optimization problem, with practical certainty the probability of satisfying state and input constraints is no less than p, with constraint violation $q_t = q_t^*$. If, on the other hand, $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then we are either in case 3.a) or 3.b), and in both cases the element $v_{k|t-k}^*$, for some $k \in [1, N-1]$, is applied to the system. Being this value part of the solution sequence $\mathcal{V}_{t-k}^*$, with corresponding constraint violation q_{t-k}^*, again the probability of satisfying state and input constraints is no less than p, with constraint violation $q_t = \tilde{q}_t = q_{t-k}^*$.

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*Politecnico di Torino, Dipartimento di Automatica e Informatica, Corso Duca degli Abruzzi 24 – 10129 Torino – Italy

**University of California at Santa Barbara, Department of Mechanical Engineering, Engineering II bldg. – 93106-5070 Santa Barbara – California

e–mail addresses: giuseppe.calafiore@polito.it, lorenzo.fagiano@polito.it

Thus, in any case, with practical certainty, at each time step the MPCS algorithm guarantees satisfaction of state and input constraints with probability no less than p and constraint violation q_t.

Proof of statement (b). Each run of Algorithm 4.1 may have one of two possible behaviors, depending on whether or not there exists a finite time $t > 0$ such that $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ and $\tilde{z}_t < d(x_t, \mathbb{X}_f)$, that is, whether or not the situation in step 3.a is ever satisfied. We then name $\mathcal{A}$ the situation when condition in step 3.a is met at some finite $t > 0$, and $\bar{\mathcal{A}}$ the complementary situation when this condition is not satisfied at any finite time, that is when $z_t^* \leq (z_{t-1} - \varepsilon d(\xi_{t-1}, \mathbb{X}_f))$ or $\tilde{z}_t \geq d(\xi_t, \mathbb{X}_f)$ holds for all $t > 0$.

I. Let us first consider the situation of case $\bar{\mathcal{A}}$. Consider a generic time t. At step 3) of the MPCS algorithm, if $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then, since it is assumed that we are in situation $\bar{\mathcal{A}}$, it must hold that $\tilde{z}_t \geq d(x_t, \mathbb{X}_f)$, thus case 3.b) occurs, and the values $\mathcal{V}_t = \tilde{\mathcal{V}}_t$ and $z_t = \tilde{z}_t$ are set. Now, recalling that $\tilde{z}_t = \max(0, z_{t-1} - d(x_{t-1}, \mathbb{X}_f))$, two cases may occur: either $\tilde{z}_t = 0$ or $\tilde{z}_t = z_{t-1} - d(x_{t-1}, \mathbb{X}_f) > 0$. If $\tilde{z}_t = 0$, we have $0 = \tilde{z}_t \geq d(x_t, \mathbb{X}_f)$, i.e. $d(x_t, \mathbb{X}_f) = 0$, which would imply that the terminal set has been reached. Otherwise, if $\tilde{z}_t = z_{t-1} - d(x_{t-1}, \mathbb{X}_f) > 0$, then we have:

$$z_t = \tilde{z}_t \geq d(x_t, \mathbb{X}_f) \geq 0, \tag{.1}$$

$$\text{and } z_t - z_{t-1} = \tilde{z}_t - z_{t-1} = z_{t-1} - d(x_{t-1}, \mathbb{X}_f) - z_{t-1} = -d(x_{t-1}, \mathbb{X}_f).$$

$$\text{Thus, } z_t - z_{t-1} \leq -\varepsilon d(x_{t-1}, \mathbb{X}_f),\ \forall x_{t-1} \notin \mathbb{X}_f, \tag{.2}$$

$$\text{and } z_t - z_{t-1} = 0 \iff x_{t-1} \in \mathbb{X}_f. \tag{.3}$$

On the other hand, if at step 3) of the MPCS algorithm it happens that $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$, then case 3.c) occurs, and the optimal values $\mathcal{V}_t^*$ and z_t^* are retained, i.e. $z_t = z_t^*$, $\mathcal{V}_t = \mathcal{V}_t^*$. In this case, it is straightforward to note that equations (.1)-(.3) still hold true. The same reasoning can be repeated for any time step, as long as the case $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ or $\tilde{z}_t \geq d(x_t, \mathbb{X}_f)$ holds true as assumed, so that we can conclude that the variable z_t enjoys the following properties:

$$\begin{aligned} & z_t \geq d(x_t, \mathbb{X}_f) \geq 0,\ \forall t \geq 0 \\ & z_t = 0 \iff x_t \in \mathbb{X}_f \\ & z_{t+1} - z_t \leq -\varepsilon d(x_t, \mathbb{X}_f),\ \forall x_t \notin \mathbb{X}_f,\ \forall t \geq 0 \\ & z_{t+1} - z_t = 0 \iff x_t \in \mathbb{X}_f. \end{aligned} \tag{.4}$$

Properties (.4) are sufficient to prove convergence of the state x_t to the set $\mathbb{X}_f$:

$$0 \leq \lim_{t\to\infty} d(x_t, \mathbb{X}_f) \leq \lim_{t\to\infty} z_t = 0, \Rightarrow \lim_{t\to\infty} d(x_t, \mathbb{X}_f) = 0.$$

Therefore, we obtain that in case $\bar{\mathcal{A}}$ the MPCS algorithm guarantees that $\lim_{t\to\infty} d(x_t, \mathbb{X}_f) = 0$.

II. Let us next analyze what happens in case $\mathcal{A}$. Let $\bar{t} > 0$ be the time instant at which the case $z_t^* > (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ and $\tilde{z}_t < d(x_t, \mathbb{X}_f)$ is met for the first time, and let $t^* < \bar{t}$ be the last time at which case $z_t^* \leq (z_{t-1} - \varepsilon d(x_{t-1}, \mathbb{X}_f))$ was satisfied, that is the last time previous to $\bar{t}$ when an optimal command sequence was retained, together with its constraint violation q_t^*, according to case 3.c) of Algorithm 4.1; let $\ell = \bar{t} - t^* \geq 1$. According to step 3.a) of the MPCS algorithm, we set

$$\mathcal{V}_{\bar{t}} = \tilde{\mathcal{V}}_{\bar{t}},\ z_{\bar{t}} = 0,\ q_{\bar{t}} = \tilde{q}_{\bar{t}}. \tag{.5}$$

Thus, at step 4) of the algorithm, the control move $u_{\bar{t}} = K_f x_{\bar{t}} + v_{0|\bar{t}}$ is applied to the system at time $\bar{t}$, where $v_{0|\bar{t}} = v^*_{\ell|t^*}$, i.e., $v_{0|\bar{t}}$ is the optimal correction predicted for time $t^* + \ell = \bar{t}$, computed at time t^*. At time step $t = \bar{t} + 1$, the state variable $x_{\bar{t}+1}$ is observed and $(\tilde{V}_{\bar{t}+1}, \tilde{z}_{\bar{t}+1}, \tilde{q}_{\bar{t}+1})$ are computed as $\tilde{z}_{\bar{t}+1} = \max(0, z_{\bar{t}} - d(x_{\bar{t}}, \mathbb{X}_f))$, $\tilde{q}_{\bar{t}+1} = q_{\bar{t}}$, $\tilde{\mathcal{V}}_{\bar{t}+1} = \{v_{1:N-1|\bar{t}}, 0\} = \{v^*_{\ell+1|t^*}, v^*_{\ell+2|t^*}, \ldots, v^*_{N-1|t^*}, 0, \ldots, 0\}$. Since (.5) holds, it must be $\tilde{z}_{\bar{t}+1} = 0$. Then, $z^*_{\bar{t}+1}$, $q^*_{\bar{t}+1}$ and $\mathcal{V}^*_{\bar{t}+1}$ are computed at step 2), and we notice that, by definition, $z^*_{\bar{t}+1} \geq 0$. Therefore, at step 3) of the algorithm either *(i)* case 3.a) $\{z^*_{\bar{t}+1} > (z_{\bar{t}} - \varepsilon d(x_{\bar{t}}, \mathbb{X}_f))$ and $\tilde{z}_{\bar{t}+1} < d(x_{\bar{t}+1}, \mathbb{X}_f)\}$ is detected again, or *(ii)* one of cases 3.b) or 3.c) are detected, which would imply, respectively, $0 = \tilde{z}_{\bar{t}+1} \geq d(x_{\bar{t}+1}, \mathbb{X}_f)$, or $0 \leq d(x_{\bar{t}+1}, \mathbb{X}_f) \leq z^*_{\bar{t}+1} = \tilde{z}_{\bar{t}+1} = 0$. Hence (in either case) $x_{\bar{t}+1} \in \mathbb{X}_f$, so that convergence to the terminal set would be achieved. Consider then case *(i)*: the values $\mathcal{V}_{\bar{t}+1} = \tilde{\mathcal{V}}_{\bar{t}+1}$, $z_{\bar{t}+1} = 0$ and $q_{\bar{t}+1} = q_{\bar{t}}$ are set in the algorithm, and the control move $u_{\bar{t}+1} = K_f x_{\bar{t}+1} + v^*_{\ell+1|t^*}$ is applied to the system. Now, the same circumstances actually reproduce for all time steps $t = \bar{t} + k$, $k \geq 0$, so the algorithm is such that the optimal input sequence $\mathcal{V}^*_{t^*}$, computed at time t^* by solving a scenario FHOCP, is the one actually next applied to the system, and the related constraint violation q_t^* is retained for all $t \geq t^*$. Thus, in case $\mathcal{A}$, there exists a finite time t^* such that the sequence $\mathcal{V}^*_{t^*}$ is applied to the system for all subsequent instants $t = t^* + k$, $k = 0, \ldots, N-1$. Now, the sequence $\mathcal{V}^*_{t^*}$ is the result of the solution of the scenario-FHOCP $\mathcal{P}(x_{t^*}, \omega_{t^*})$, and Proposition 3.1 states that, with practical certainty, we have $R(\omega_{t^*}) \geq p$, where R is the reliability defined in Section III-A of the paper, which means that $\mathbb{P}\{\delta : h(s^*_{t^*}, x_{t^*}, \delta) \leq 0\} \geq p$. Therefore, in the situation $\mathcal{A}$, there exists a finite time t^* at which an optimal control sequence is computed by solving a scenario-FHOPC and next applied to the actual system for the subsequent N time instants: we can hence claim with practical certainty this sequence will satisfy the problem constraints and reach the terminal set within the time window from t^* to $t^* + N$, with probability at least p and constraint violation q_t^*.