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Set Membership approximation of discontinuous Nonlinear Model Predictive Control laws ⋆

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Abstract

In this paper, the use of Set Membership (SM) methods is investigated, in order to derive off-line an approximation of a discontinuous nonlinear model predictive control (NMPC) law. The approximating function can then be evaluated on-line, instead of solving the nonlinear program embedded in the NMPC scheme. This way, a significant decrease of computational times may be obtained, thus allowing the application of NMPC also to systems with "fast" dynamics. It is shown that the knowledge of the discontinuities is needed to achieve an approximated controller with arbitrarily small approximation error. By exploiting such a knowledge, SM techniques already developed for the case of continuous NMPC laws are generalized in order to approximate discontinuous ones. The stability of the origin of the closed loop system with the approximated control law is analyzed, and a numerical example is employed to illustrate the features of the proposed approach.

Key words: Predictive control; Function approximation; Discontinuous control; Nonlinear systems; Set Membership approximation theory.

1 Introduction

In Nonlinear Model Predictive Control (NMPC, see e.g. [17]) the control action is computed by means of a Receding Horizon (RH) strategy, which requires at each sampling time the solution of a nonlinear program (NLP, see e.g. [22]), where the systems state x (and, possibly, other measured parameters and reference variables) is a parameter in the optimization. For time invariant systems, the solution of the NLP defines a static nonlinear function $\kappa(x)$, denoted in this paper as the "nominal" control law. In the last decade, a significant research effort has been devoted to the problem of efficient implementation of NMPC laws, motivated by the objective of applying this control strategy also to systems with relatively "fast" dynamics, in which the employed sampling period does not allow the real-time solution of the NLP. A possible viable approach is the use of an approximated NMPC law $\hat{\kappa} \approx \kappa$, derived off-line and then evaluated on-line instead of solving the NLP. Contributions in the field of approximated NMPC can be found e.g. in [23,11,28,3,4,9,27,5,25], using various approaches. In particular, approximation techniques based on Set Membership (SM) theory have been developed and studied in [3–5]. In this framework, approximated NMPC laws with guaranteed accuracy (in terms of a bound on the pointwise error $\kappa(x)-\hat{\kappa}(x)$) and consequent performance and stability properties have been derived, with the assumption of continuity of κ over the compact subset $\mathcal{X}$ of the state space considered for the approximation. Although the assumption of continuity of κ holds for MPC with linear and "almost linear" models [18] and for a series of problems with nonlinear models and/or nonlinear constraints, it is well-known that NMPC laws may be discontinuous and that there exist systems that cannot be stabilized with continuous control laws (see e.g. [21,19,20,13]). In these cases, the guaranteed properties of the existing SM approaches do not hold anymore. In the described context, the contributions of this paper are a) to show through a motivating example that the knowledge of the discontinuities is needed in order to achieve an arbitrarily small pointwise approximation error, which is required to retain the closed-loop stability properties, b) to use such a knowledge to derive an approximation of a discontinuous NMPC law, using a SM approach, and c) to study the closed loop system stability when the SM approximated law is used. Finally, a numerical example is employed to show the features of the proposed approximation technique.

2 Problem settings

Consider the following nonlinear state space model:

$$x_{t+1} = f(x_t, u_t) + w_t, \ \ \|w_t\|_2 \leq \mu \tag{1}$$

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where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system state and control input, respectively, and $w_t \in \mathbb{R}^n$ is an unknown but bounded disturbance.

Assumption 1 *Function f in (1) is continuous with respect to u_t, i.e. for any fixed value $\overline{x}_t$ the function $\overline{f}(u_t) = f(\overline{x}_t, u_t)$ is continuous over $\mathbb{R}^m$.*

The control objective is to regulate the system state to the origin under some input and state constraints, represented by a convex set $\mathbb{X} \subseteq \mathbb{R}^n$ and a compact set $\mathbb{U} \subseteq \mathbb{R}^m$, both containing the origin in their interiors, in which the state and input values x_t and u_t should be kept, respectively. In NMPC, the control law $u_t = \kappa(x_t)$ is defined implicitly by a RH strategy, in which at each sampling instant the following NLP has to be solved:

$$\min_U \quad J(U, x_t) \doteq \sum_{k=0}^{N_p-1} L(x_{t+k|t}, u_{t+k|t}) + \Phi(x_{t+N_p|t}) \tag{2a}$$

s. t.

$$x_{t+k|t} \in \mathbb{X},\ k = 1, \ldots, N_p \tag{2b}$$

$$u_{t+k|t} \in \mathbb{U},\ k = 0, \ldots, N_p \tag{2c}$$

$$\textit{Stabilizing constraints} \tag{2d}$$

where $x_{t+k|t}$ denotes k steps ahead state prediction using the model (1), given the input sequence $u_{t|t}, \ldots, u_{t+k-1|t}$ and the "initial" state $x_{t|t} = x_t$, and $U = \left[u_{t|t}^T, \ldots, u_{t+N_c-1|t}^T\right]^T$ is the vector of the control moves to be optimized. $N_p \in \mathbb{N}$ and $N_c \in \mathbb{N}$, $N_c \leq N_p - 1$ are the prediction and the control horizons, respectively. The remaining predicted control moves $[u_{t+N_c|t}, \ldots, u_{t+N_p-1|t}]$ can be computed with different strategies [17]. The optimal cost and its optimizer are indicated as $J^*(U^*(x_t), x_t)$ and $U^*(x_t)$. It is assumed that the optimization problem (2) is feasible over a set $\mathcal{F} \subseteq \mathbb{R}^n$ which will be referred to as the "feasibility set", so that $\kappa : \mathcal{F} \to \mathbb{U}$. The application of the RH controller gives rise to the following nonlinear state feedback configuration:

$$x_{t+1} = f(x_t, \kappa(x_t)) + w_t = F^0(x_t) + w_t = F^0_w(x_t, w_t) \tag{3}$$

The system (3) will be also referred to as the "nominal" closed loop system in the following. The set of solutions of (3) at the generic time instant t, starting from the initial condition $x_0 \in \mathcal{F}$ and considering all the possible realizations of the disturbance sequences $\{w\}$, is indicated here as $\mathcal{S}^0_\mu(t, x_0) \doteq \{\phi^0_w(t, x_0) = \underbrace{F^0_w(F^0_w(\ldots F^0_w(x_0, w_0)))}_{t \text{ times}},$ $\forall \{w_k\} : \|w_k\|_2 \leq \mu,\ k = 0, \ldots, t\}$. With a proper choice of the cost function J and of the stabilizing constraints (2d), it is possible to achieve robust closed loop stability and constraint satisfaction properties in the presence of the disturbance w. In particular, existing approaches for robust NMPC exploit constraint tightening, state contraction constraints, terminal set constraints, min-max formulations and input-to-state stability (ISS) techniques (see e.g. [17,8,6,16,14,24]). In this paper, κ will be approximated over a compact set $\mathcal{X} \in \mathcal{F}$. An important theoretical issue to be addressed in the approximation of κ concerns the capability to provide a guaranteed approximation accuracy and the evaluation of the effects of the approximation on the closed loop stability properties. In [3–5] it has been shown that, in the context of SM approximation theory, it is possible to derive an approximated control law κ^{SM}, based on the preliminary off-line computation of a finite number ν of nominal control moves, that enjoys the following three properties:

$$\kappa^{\mathrm{SM}} : \mathcal{X} \to \mathbb{U} \tag{4a}$$

$$|\kappa(x) - \kappa^{\mathrm{SM}}(x)| \leq \zeta < \infty,\ \forall x \in \mathcal{X} \tag{4b}$$

$$\lim_{\nu \to \infty} \zeta = 0 \tag{4c}$$

Properties (4a)-(4c), namely satisfaction of input constraints, bounded approximation accuracy with a finite bound ζ and convergence of ζ to zero, have been proved to be sufficient to be able to achieve closed loop stability and guaranteed regulation precision when the function κ^{SM} is employed for feedback control [3]. However, the above-mentioned results rely on the assumption of continuity of κ in addition to its stabilizing properties, while it is known that, depending on the system model, the constraints and the chosen objective function, the nominal NMPC law may be discontinuous. In this case, in general, it is not possible to achieve the key property (4c), unless some other information on the discontinuities of κ is available. Moreover, the absence of property (4c) may lead to instability or to a reduction of the region of attraction for the origin of the closed-loop system. This concept will be illustrated in the next Section, through a motivating example.

3 A motivating example

Consider the following system model:

$$x_{t+1} = a\, x_t + [b\, s(x_t) + c]\, u_t \tag{5}$$

where x_t, $u_t \in \mathbb{R}$, $s(x_t) = -1$ if $x_t < 1$ and $s(x_t) = 1$ if $x_t \geq 1$. Consider the NLP (2) with $U = u_{t|t}$, cost function $J(U, x_t) = x^2_{t+1|t} + R\, u_t^2$, $R > 0$, and the stabilizing constraint $g(x_t, u_t) \leq 0$, with $g(x_t, u_t) = x^2_{t+1|t} - \alpha\, x_t^2$, $\alpha \in (0, 1)$ (contraction constraint). The contraction constraint ensures closed loop stability and it can be easily noted that a Lyapunov function for the closed loop system is $V(x) = x^2$. For any given value of x_t, the cost and constraint functions are convex (quadratic) functions of u_t. Thus, the Karush-Kuhn-Tucker (KKT) conditions (see e.g. [22]) are sufficient for global optimality. Assuming that $a = 5$, $b = 1$, $c = 0.5$, $R = 2$, $\alpha = 0.81$, by imposing the KKT conditions the following explicit optimal control law is obtained:

$$\kappa(x_t) = -\frac{4.1}{s(x_t) + 0.5} x_t \tag{6}$$

Function κ (6) is clearly discontinuous at $x_t = 1$, however let us assume that this information is not available for the approximation. Assume now that the approximation of κ has to be carried out on the set $\mathcal{X} = [-0.5, 1.5]$, and consider the following $\nu = 8$ nominal control values as part of the prior information on κ: $\tilde{u} = \{\kappa(\tilde{x}) : \tilde{x} \in \mathcal{X}_\nu\}$, where $\mathcal{X}_\nu =$

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$\{-0.50, -0.24, 0.02, 0.28, 0.54, 0.80, 1.06, 1.32\}$. Moreover, consider the Nearest Point (NP) approximation (see e.g. [3]) of function (6), i.e. $\kappa^{\text{NP}}(x) = \kappa(\tilde{x}^{\text{NP}})$, where $\tilde{x}^{\text{NP}} = \arg\min_{\tilde{x}\in\mathcal{X}_\nu} \|x - \tilde{x}\|_2$. The nominal and approximated control laws are shown in Fig. 1. In this case, the worst-case error

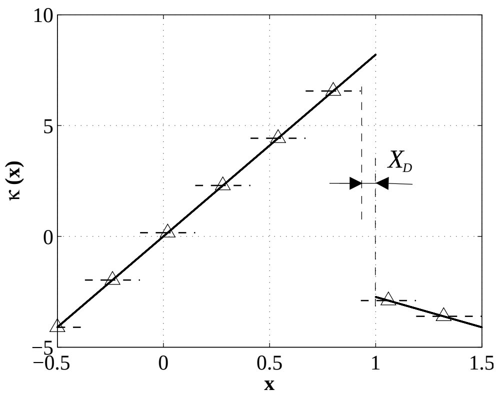

Fig. 1. Motivating example: discontinuous NMPC law approximation without considering any information on the discontinuity. Nominal function $\kappa(x)$ (solid) and NP approximation (dashed). Off-line-computed exact control moves indicated with triangles.

bound $\zeta = \sup_{x\in\mathcal{X}} \|\kappa(x) - \kappa^{\text{NP}}(x)\|_2$ is approximately equal to 10.93 and its convergence to zero as ν increases (i.e. property (4c)) is not achieved, in the sense that there will always be a subset $X_D(\nu) \subset \mathcal{X}$ (e.g. the set $X_D = (0.93, 1)$ in Fig. 1) with "high" approximation error. Similar considerations hold for any other approximation method (e.g. neural networks, linear interpolation, SM optimal approximation [3],...). In fact, as long as no prior knowledge on the discontinuity is taken into account, it is not possible to achieve arbitrarily small pointwise approximation error and, more in general, to derive a "good" approximation (this concept is well-known in standard mathematical analysis, one may think about the Gibbs phenomenon for Fourier series, see e.g. [7]). Moreover, through this motivating example it can be shown that the absence of property (4c) may in general hinder the possibility to guarantee trajectory boundedness and convergence over all $\mathcal{X}$ when the approximated control law is used, even if a very large number of off-line computed points is considered. This aspect can be evidenced by simulating the closed loop state trajectory obtained with κ and the one obtained with κ^{NP}, using as an example $\nu = 2000$ off-line computed data (chosen with uniform gridding over $\mathcal{X}$) and starting from the initial state $x_0 = 0.999$: it can be noted that with the approximated control law the state trajectory diverges and it exits the set $\mathcal{X}$ already after one simulation step. This issue occurs in particular when the origin is an unstable fixed point in open loop, like in this example. In summary, the simple numerical example given in this Section motivates the need for some information on the discontinuities, in order to still be able to achieve the property (4c) and, consequently, closed loop stability. In the next Section, it will be specified what type of prior information on the discontinuities is assumed to be known in this paper, and how such a knowledge can be employed to derive an approximation of a discontinuous NMPC law κ, enjoying the properties (4).

4 SM approximation of discontinuous NMPC

With the SM approach, the approximated NMPC law $\kappa^{\text{SM}} \approx \kappa$ is derived on the basis of the available *prior information* on κ, which will be now resumed. For the sake of simplicity and without loss of generality, in the following it is considered that $u \in \mathbb{R}$ (i.e. the system f is single-input). Multiple inputs can be treated by approximating each component κ_i, $i = 1, \ldots, m$, separately.

4.1 Prior information

The available information on κ includes the knowledge of a finite number ν of exact control moves $\tilde{u}$, computed off-line and stored:

$$\tilde{u}^k \doteq \kappa(\tilde{x}^k), k = 1, \ldots, \nu \tag{7}$$

The state values $\tilde{x}^k$ define the set

$$\mathcal{X}_\nu \doteq \{\tilde{x}^k,\ k = 1, \ldots, \nu\} \subseteq \mathcal{F}. \tag{8}$$

It is assumed that $\mathcal{X}_\nu$ is chosen in such a way that the following property holds:

$$\lim_{\nu\to\infty} d(\mathcal{X}, \mathcal{X}_\nu) = 0 \tag{9}$$

where $d(\mathcal{X}, \mathcal{X}_\nu)$ is defined as $d(\mathcal{X}, \mathcal{X}_\nu) \doteq \sup_{x\in\mathcal{X}} \min_{\tilde{x}\in\mathcal{X}_\nu} (\|x - \tilde{x}\|_2)$ and it is equivalent to the Hausdorff distance between sets $\mathcal{X}$ and $\mathcal{X}_\nu$, see e.g. [2]. For example, uniform gridding over $\mathcal{X}$ satisfies condition (9). Moreover, as anticipated in Section 2, information on the discontinuities of κ is also assumed to be available. In view of the motivating example shown in Section 3, such information is needed to be able to satisfy property (4c). Knowledge on the discontinuities can be obtained either from the structure of the problem (see e.g. the examples of Sections 3 and 5), or by using numerical methods for function approximation from data (see e.g. [1,12,26], which employ basis functions like polynomials, radial basis functions or wavelets/shearlets). To be more specific, the information on the regularity of κ, which is assumed to be available in this paper, can be resumed as follows:

Assumption 2 $\exists X^j \subset \mathcal{F},\ j = 0, \ldots, r < \infty$ *such that* ***i)*** $\mathcal{X} \subset \bigcup_{j=0}^{r} X^j$, ***ii)*** $X^j \bigcap X^{i\neq j} = \varnothing$, $j = 0, \ldots, r$, *and* ***iii)*** κ *is continuous on* X^j, $j = 0, \ldots, r$

Assumption 2 implies that the points $x \in \mathcal{X}$ where κ is discontinuous may lie only on the boundaries of the sets X^j. Now, on the basis of Assumption (2) and of the boundedness of the image set $\mathbb{U}$ of κ, it can be noted that κ is Lipschitz continuous over each one of the sets X^j, $j = 0, \ldots, r$, with corresponding Lipschitz constants L^j_κ. Moreover, the set $\mathcal{X}_\nu$ (8) can be partitioned in a finite number r of subsets $\mathcal{X}^j_\nu$:

$$\mathcal{X}^j_\nu = \mathcal{X}_\nu \bigcap X^j,\ j = 0, \ldots, r \tag{10}$$

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The finite number of off-line computed values contained in the generic set $\mathcal{X}_\nu^j$ is indicated in the following as ν^j. Clearly, it holds that $\sum_{j=0}^{r} \nu^j = \nu$. Thus, the overall prior information on κ can be resumed as follows:

$$\kappa \in FFS \tag{11}$$

where FFS is the Feasible Function Set, defined as:

$$FFS \doteq \{\kappa : \mathcal{X} \to \mathbb{U},\ |\kappa(x^1) - \kappa(x^2)| \le L_\kappa^j \|x^1 - x^2\|_2,\ \forall x^1, x^2 \in X^j,\ \kappa(\tilde{x}^k) = \tilde{u}^k,\ k = 1, \dots, \nu\} \tag{12}$$

Estimates $\hat{L}_\kappa^j$, $j = 0, \dots, r$ of the constants L_κ^j, $j = 0, \dots, r$ with guaranteed convergence to the actual values can be obtained as shown in [3].

4.2 SM approximation of discontinuous NMPC

On the basis of the considered prior information (11)-(12), SM techniques already presented e.g. in [3–5] can be employed on each partition X^j, in order to derive approximating functions $\kappa^{\mathrm{SM},j}$ with arbitrary small guaranteed approximation error bound $\zeta^j(\nu^j)$. Note that in general a different SM technique can be used on each partition, in order to adapt to the properties of data. It can be proved (see e.g. [3]) that each one of the functions $\kappa^{\mathrm{SM},j}$ enjoys the properties (4) on its own partition:

$$\kappa^{\mathrm{SM},j} : X^j \to \mathbb{U} \tag{13a}$$

$$|\kappa(x) - \kappa^{\mathrm{SM},j}(x)| \le \zeta^{\mathrm{SM},j} < \infty,\ \forall x \in X^j \tag{13b}$$

$$\lim_{\nu \to \infty} \zeta^{\mathrm{SM},j}(\nu) = 0 \tag{13c}$$

If an approximated control law $\kappa^{\mathrm{SM},j}$ is computed for each region X^j, the following approximating function can be defined:

$$\forall x \in \mathcal{X},\ \kappa^{\mathrm{SM}}(x) \doteq \kappa^{\mathrm{SM},j^*}(x),\ \text{with}\ j^* \in [0, r] : x \in X^{j^*} \tag{14}$$

Then, the following result is obtained:

Lemma 1 *For any $x \in \mathcal{X}$ the approximation $\kappa^{SM}(x)$ enjoys the following properties:*

$$\kappa^{SM} : \mathcal{X} \to \mathbb{U} \tag{15a}$$

$$|\kappa(x) - \kappa^{SM}(x)| \le \zeta \doteq \max_j \zeta^{SM,j},\ \forall x \in \mathcal{X} \tag{15b}$$

$$\lim_{\nu^j \to \infty} \zeta = 0 \tag{15c}$$

Proof. For any $x \in \mathcal{X}$, on the basis of Assumption 2 the value of j^* such that $x \in X^{j^*}$ exists and is unique. Then, the properties (15) are direct consequences of the properties (13) of the SM approximation employed over X^{j^*}, considering that by definition $\zeta^{\mathrm{SM},j^*} \le \zeta$ and that $\lim_{\nu^j \to \infty} \zeta = \lim_{\nu^j \to \infty} \max_j \zeta^{\mathrm{SM},j} = 0$. □

Lemma 1 implies that the overall guaranteed approximation error can be made arbitrarily small by increasing ν^j: this property is needed for the stability analysis given in Section 4.3.

Thus, at each sampling instant t the implementation algorithm of κ^{SM} is the following:

1. get x_t at time instant t
2. find j^* such that $x_t \in X^{j^*}$ (16)
3. compute $u_t = \kappa^{\mathrm{SM},j^*}(x_t)$

4.3 Closed loop stability result

The use of control law κ^{SM} gives rise to the following state feedback configuration:

$$x_{t+1} = f(x_t, \kappa^{\mathrm{SM}}(x_t)) + w_t = F_w^{\mathrm{SM}}(x_t, w_t),\ \|w_t\|_2 \le \mu. \tag{17}$$

The aim is to investigate the stability properties of system (17), considering the guaranteed accuracy features of κ^{SM} (given by Lemma 1) and the stability properties of the nominal closed loop system (3). To this end, the following Assumption on system (3) is considered:

Assumption 3 *The nonlinear autonomous system (3) is ISS stable in $\mathcal{X}$ for any $w_t : \|w_t\|_2 \le \overline{\mu}$, where $\overline{\mu} > \mu$*

As it will be clear later on in this Section, the reason why a higher disturbance bound $\overline{\mu}$ with respect to the bound μ in (1) is considered in Assumption 3 is related to the effects, on the closed loop system, of the approximation errors that occur when the nominal NMPC law is replaced by the approximated one. Assumption 3 can be met by using the robust synthesis techniques for NMPC mentioned in Section 2. ISS definition is recalled below (see e.g. [15]).

Definition 1 *Given a compact set $\mathcal{X}$, including the origin in its interior, system (3) with $\|w\|_2 \le \overline{\mu}$ is ISS stable in $\mathcal{X}$ if **i)** the set $\mathcal{X}$ is positively invariant for any trajectory $\phi_w^0(t,x) \in \mathcal{S}_{\overline{\mu}}^0(t,x)$: $\mathcal{S}_{\overline{\mu}}^0(t,x) \subset \mathcal{X}$, $\forall t \ge 0$, $\forall x \in \mathcal{X}$, and **ii)** there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that $\|\phi_w^0(t,x)\|_2 \le \beta(\|x\|_2, t) + \gamma(\overline{\mu})$, $\forall x \in \mathcal{X}$, $\forall \phi_w^0(t,x) \in \mathcal{S}_{\overline{\mu}}^0(t,x)$.*

Recall that a function β is said to be a $\mathcal{KL}$-function if, for each $t \in \mathbb{N}$, $\beta(\cdot, t)$ is non-decreasing and $\lim_{t \to \infty} \beta(s,t) = 0$ for each $s \ge 0$. A function γ is said to be a $\mathcal{K}$-function if it is continuous, zero at zero, and strictly increasing. Thus, on the basis of Assumption 3, the trajectory of the closed loop system (3) converges, as $t \to \infty$, to a neighborhood of the origin, described by the set $B_{\gamma(\mu)}$, defined as:

$$B_{\gamma(\mu)} = \{x : \|x\|_2 \le \gamma(\mu)\}. \tag{18}$$

The following result is derived:

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Theorem 1 *Let Assumptions 1-3 hold and consider the closed loop system (17), where the control law κ^{SM} is a SM approximation of a discontinuous nominal NMPC law κ, computed as defined in (14) by using the prior information (11). Then, it is always possible to compute a finite number ν of exact control moves such that* ***i)*** *the set $\mathcal{X}$ is positively invariant for system (17), and* ***ii)*** *the trajectories of system (17) converge to an arbitrarily small neighborhood of the set $B_{\gamma(\mu)}$ (18), for any initial condition $x_0 \in \mathcal{X}$.*

Proof. For any fixed value $\overline{x}$ of x, by Assumption 1 $f(\overline{x}, u)$ is continuous with respect to u, which belongs to $\mathbb{U}$. Since $\mathbb{U}$ is compact, $f(\overline{x}, u)$ results to be Lipschitz continuous over $\mathbb{U}$ with constant $\overline{L}_f(\overline{x})$. Then, for any $x \in \mathcal{X}$, on the basis of Lemma 1 (property (15b)) a bound on the one-step trajectory perturbation between systems $F^0_w(x, w)$ and $F^{\text{SM}}_w(x, w)$ can be obtained as $\|F^0(x, w) - F^{\text{SM}}(x, w)\|_2 = \|f(x, \kappa(x)) + w - f(x, \kappa^{\text{SM}}(x)) - w\|_2 \leq L_f|\kappa(x) - \kappa^{\text{SM}}(x)| \leq L_f\zeta \Rightarrow \|F^0_w(x, w) - F^{\text{SM}}_w(x, w)\|_2 \leq \mu^{\text{SM}} \doteq L_f\zeta$, where $L_f = \sup_{\overline{x}\in\mathcal{X}} \overline{L}_f(\overline{x}) < \infty$. Thus, the effects of the approximation error can be treated as a further additive disturbance v, i.e. $F^{\text{SM}}_w(x, w) = F^0_w(x, w) + v = F^0_w(x, w + v)$, $\|w + v\|_2 \leq \|w\|_2 + \|v\|_2 \leq \mu + \mu^{\text{SM}}$. Moreover, on the basis of Lemma 1 (property (4c) it can be noted that $\lim_{\nu\to\infty} \left(\mu + \mu^{\text{SM}}\right) = \mu + L_f \lim_{\nu\to\infty} \zeta = \mu$. Thus, by increasing ν, the worst-case bound on the overall additive perturbation $w + v$ gets arbitrarily close to μ. Indeed, considering the Assumption 3, it is sufficient that $\mu^{\text{SM}} \leq \overline{\mu} - \mu$ to obtain positive invariance of the set $\mathcal{X}$ and convergence of the closed loop trajectories to the set $\{x : \|x\|_2 \leq \gamma(\mu + \mu^{\text{SM}})\}$. Moreover, since γ is a class-$\mathcal{K}$ function, trajectory convergence to an arbitrary small neighborhood of the set $B_{\gamma(\mu)}$ is achieved, since $\lim_{\nu\to\infty} \gamma(\mu + \mu^{\text{SM}}) = \gamma(\mu)$ □

4.4 Motivating example revisited

In the example of Section 3, by taking into account the discontinuity of κ, a possible choice of X^j satisfying Assumption 2 is $X^0 = [-0.5, 1)$, $X^1 = [1, 1.5]$. By using the NP approach on each region, the SM approximation κ^{SM} (14) can be computed as:

$$\kappa^{\text{SM}}(x) \doteq \begin{cases} \kappa^{\text{NP},0}(x), \text{ for } x \in X^0 \\ \kappa^{\text{NP},1}(x), \text{ for } x \in X^1 \end{cases} \tag{19}$$

where $\kappa^{\text{NP},0}$ and $\kappa^{\text{NP},1}$ are the NP approximations computed over the regions X^0, X^1, respectively, using the related sets $\mathcal{X}^0_\nu$ and $\mathcal{X}^1_\nu$ (10) of state values considered in the off-line computations. On each region, function κ (6) is Lipschitz continuous, with constants $L^0_\kappa = 8.2$ and $L^1_\kappa = 2.733$ corresponding to regions X^0 and X^1, respectively, while the values of $d(X^0, \mathcal{X}^0_\nu)$ and $d(X^1, \mathcal{X}^1_\nu)$ are equal to 0.2 and 0.18, respectively. Thus, the worst case approximation errors obtained with the NP technique on each region are bounded by $\zeta^0 = L^0_\kappa d(X^0, \mathcal{X}^0_\nu) = 1.64$ and $\zeta^1 = L^1_\kappa d(X^1, \mathcal{X}^1_\nu) = 0.5$

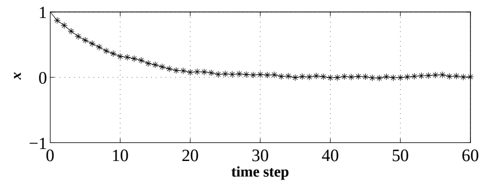

Fig. 2. Motivating example revisited: closed loop state trajectory obtained with a SM approximation (with $\nu = 200$) of the discontinuous NMPC law, taking into account the knowledge of the discontinuity. Initial state: $x_0 = 0.999$

(see e.g. [3] for details on the guaranteed bounds with the NP approach), and the overall worst-case approximation error ζ (15b) is $\zeta = \max(\zeta^0, \zeta^1) = 1.64$. If ν is increased in such a way that (9) holds, the values of $d(X^0, \mathcal{X}^0_\nu)$ and $d(X^1, \mathcal{X}^1_\nu)$ decrease and consequently $\zeta(\nu)$ can be made arbitrarily small. Such a condition, together with the existence of a continuous Lyapunov function for the system controlled with the nominal NMPC law, is sufficient to be able to achieve trajectory boundedness and convergence to an arbitrarily small neighborhood of the origin, as it can be proved with little modifications with respect to the results of [3]). Fig. 2 depicts the closed loop state trajectory obtained with κ^{SM} (19), using $\nu = 200$ off-line computed data (chosen with uniform gridding over $\mathcal{X}$) and starting from the initial state $x_0 = 0.999$: it can be noted that with the approximated control law the state trajectory now converges to a neighborhood of the origin.

5 Numerical example

Consider the discrete-time system:

$$\begin{cases} x_{1,t+1} = 1.01x_{1,t} + 0.1x_{2,t} + w_{1,t} \\ x_{2,t+1} = x_{2,t} + \cos(x_{1,t})u_t + w_{2,t} \end{cases} \tag{20}$$

The additive disturbance $w_t \in \mathbb{R}^2$ is bounded as $\|w_t\|_2 \leq 0.05$, $\forall t$ (i.e. $\mu = 0.05$). The constraint sets are $\mathbb{U} = \{u \in \mathbb{R} : |u| \leq 5\}$ and $\mathbb{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 3\}$. The origin is an open-loop unstable fixed point for system (20) and the system dynamics are “almost linear” in its neighborhood (i.e. when $\cos(x_{1,t}) \simeq 1$). The nominal NMPC law κ has been designed according to (2) with prediction and control horizons $N_p = N_c = 35$, $L(x, u) = 3x_1^2 + 20x_2^2$ and $\Phi(x) = 3x_1^2 + 20x_2^2$. In order to enforce nominal stability, the following terminal state constraint has been employed:

$$x_{t+N_p|t} = 0. \tag{21}$$

Note that no weights on the input variable have been included in the cost function. This choice, together with the presence of the nonlinearity in (20), gives rise to discontinuities in the nominal control law. In fact, the closer the value of $\cos(x_{1,t})$ to zero, the higher the control input magnitude (since there is no weight on the input), whose sign changes when $\cos(x_{1,t})$ crosses the zero value. Thus, the nominal control law results to be discontinuous at all the points that belong to the set $D = \{x \in \mathbb{R}^2 : x_1 = \frac{\pi}{2} + q\pi,\ q \in \mathbb{Z}\} \bigcap \mathcal{F}$.

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These discontinuities are evidenced in Fig. 3 (left), where the level curves of the nominal control law κ are shown together with the considered set $\mathcal{X}$. A possible choice of partitions that satisfies Assumption 2 is (see Fig. 3, left):

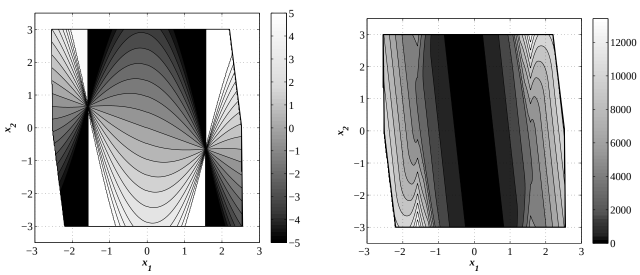

Fig. 3. Numerical example: set $\mathcal{X}$ considered for the approximation and level curves of the nominal control law κ (left) and of the Lyapunov function $J^*(U^*(x), x)$.

$X^0 = \{x \in \mathcal{X} : x_1 < -\frac{\pi}{2}\}$, $X^1 = \{x \in \mathcal{X} : -\frac{\pi}{2} < x_1 < \frac{\pi}{2}\}$, $X^2 = \{x \in \mathcal{X} : x_1 > \frac{\pi}{2}\}$. Note that, in this example, the discontinuity of the control law occurs when the state variable x is such that $f(x, u) = f(x)$, i.e. when the control input does not influence the system dynamics: therefore, it is not useful to try to approximate the nominal control law when $x \in D$. This is the reason why the state values $\{x : |x_1| = \frac{\pi}{2}\}$ have not been included in the partitions X^0, X^1, X^2.

By numerical inspection, the optimal cost function $J^*(U^*(x), x)$, reported in Fig. Fig. 3 (right), results to be continuous over $\mathcal{X}$. Thus, function $J^*(U^*(x), x)$ is a continuous Lyapunov function (thanks to the stabilizing constraint (21)) for the nominal closed loop system, which is therefore locally Input-to-State-Stable (see e.g. [10]). The SM control law (14) has been derived by using, on each partition, a NP approximation. The search for the active partition (see the algorithm (16)) is straightforward in this example, since a simple inequality check on the state variable x_1 is required. Fig. 4 shows the trajectory of the system state using either the nominal control law or the approximated one, with $\nu = 10^3$ and with initial condition $x_0 = [-1.25, -3]^T$. A uniform gridding over $\mathcal{X}$ has been used to compute off-line the control moves (7). It can be noted that the state trajectories obtained with κ^{SM} are practically superimposed to the ones obtained with κ while they are "far" from the origin (i.e. in the first 3-4 time steps). Both the nominal controller and the approximated one are able to regulate the system state to a neighborhood of the origin, contained in the ball sets $B_{0.34} = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 0.34\}$ and $B_{0.48} = \{x \in \mathbb{R}^2 : \|x\|_2 \leq 0.48\}$, respectively (see Fig. 4). Better regulation precision can be obtained either by increasing the number of off-line computed nominal control moves close to the origin, or by using other approximation methods, like the local optimal SM approach [4] or the techniques studied in [5]. As regards the computational times, after extensive simulations starting from random initial conditions inside $\mathcal{X}$, the maximum and average computational times of the on-line optimization resulted to be 31 s and 2×10^{-1} s, respectively, using Matlab® 7 with an

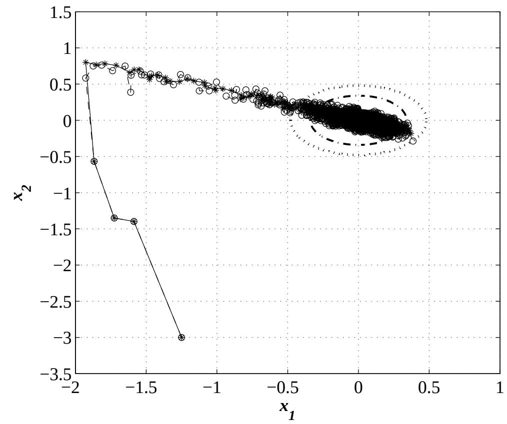

Fig. 4. Numerical example: (state trajectory obtained with the nominal (solid line with "*" marker) and approximated (dashed line with "○" marker) control laws, and sets $B_{0.34}$ (dash-dotted line), $B_{0.48}$ (dotted line) to which the state trajectories converge, respectively. Initial state $x_0 = [-1.25, -3]^T$. SM approximation carried out with $\nu = 10^3$.

Intel® Core™2 Duo processor at 2.4 GHz and 2 GB RAM. The maximal and average computational times achieved with the SM approximation, with $\nu = 10^3$, are equal to 3.8×10^{-4} s and 3.6×10^{-4} s, respectively. Tradeoffs between computational times and accuracy can be achieved by applying different approximation approaches [5]. Indeed, the reported computational times are intended to be used for relative comparisons only, since their values depend on the employed hardware.

6 Conclusions

In this paper, the existing SM approaches [3–5] for approximate NMPC have been generalized to the case of discontinuous NMPC. The main drawback of the presented results is that the procedure to derive the SM approximation is not as systematic as in the continuous case, due to the need of finding the discontinuities of the nominal controller. On the other hand, it has been shown that the prior information on the discontinuities is needed in general, in order to achieve guaranteed and arbitrarily small approximation error, and that the absence of the latter property may hamper the closed loop stability properties.

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