Set Membership approximation theory for fast implementation of Model Predictive Control laws ⋆

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Abstract

In this paper, the use of Set Membership (SM) methodologies is investigated in the approximation of Model Predictive Control (MPC) laws for linear systems. Such approximated MPC laws are derived from a finite number ν of exact control moves computed off-line. Properties in terms of guaranteed approximation error, closed-loop stability and performance are derived assuming only the continuity of the exact predictive control law. These results are achieved by means of two main contributions. At first, it will be shown that if the approximating function enjoys two *key properties* (i.e. fulfillment of input constraints and explicit evaluation of a bound on the approximation error, which converges to zero as ν increases), then it is possible to guarantee the boundedness of the controlled state trajectories inside a compact set, their convergency to an arbitrary small neighborhood of the origin, and satisfaction of state constraints. Moreover, the guaranteed performance degradation, in terms of maximum state trajectory distance, can be explicitly computed and reduced to an arbitrary small value, by increasing ν. Then, two SM approximations are investigated, both enjoying the above *key properties*. The first one minimizes the guaranteed approximation error, but its on-line computational time increases with ν. The second one has higher approximation error, but lower on-line computational time which is constant with ν when the off-line computed moves are suitably chosen. The presented approximation techniques can be systematically employed to obtain an efficient MPC implementation for "fast" processes. The effectiveness of the proposed techniques is tested on two numerical examples.

Key words: Function approximation, Predictive control, Constraint satisfaction, Nonlinear systems analysis

1 Introduction

Model Predictive Control (MPC) (see e.g. the survey of Mayne *et al.*, 2000) is a model based control technique where the control action is computed by means of a receding horizon strategy, which requires at each sampling time the solution of an optimization problem. For time invariant systems, it results that the input u is a nonlinear static function of the system state x, i.e. $u_t = f^0(x_t)$. The receding horizon strategy leads to strong limitations in using MPC techniques in the presence of fast plant dynamics which require small sampling periods that do not allow to perform the optimization problem on-line. In order to use MPC in a larger range of applications, a significant research effort has been devoted in recent years to the problem of efficient implementation of MPC laws. One line of research is the development of more efficient techniques to solve the optimization problem on-line (see e.g. Diehl and Björnberg, 2004; Alamir, 2006). A second line of research is to investigate the derivation of explicit expression of f^0 to be used for the on-line implementation. This is the case of MPC for (piecewise) linear systems with linear constraints and quadratic cost function, where it has been shown (see Bemporad *et al.*, 2002; Seron *et al.*, 2003; Borrelli *et al.*, 2005) that f^0 is a piecewise affine (PWA) function defined on a polyhedral partition of the state space, which can be computed off-line and stored. However, since the number of such polyhedral regions increases significantly with the state dimension and the control horizon length, severe limitations may occur in the on-line computation of the control move, due to the computational time needed to find the partition the actual state lies in. Thus, other methodologies for exact explicit MPC have been introduced: the construction of a binary search tree to evaluate the PWA control law has been described by Tondel *et al.* (2002), achieving logarithmic computational time in the number of regions, while Johansen *et al.* (2002) and Bemporad and Filippi (2003) developed explicit suboptimal solutions, with lower numbers of regions.
A third line of research on fast MPC implementation relies on the on-line evaluation of an approximated control law $\widehat{f} \approx f^0$, computed on the basis of a finite number ν of exact control moves, evaluated off-line. Such an approach can be used also in cases where an explicit expression of f^0 is

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not available, as it happens e.g. if nonlinear constraints or non-quadratic cost functions are considered.

A first contribution along this line was given by Parisini and Zoppoli (1995), using a neural network approximation of f^0. However, no guaranteed approximation error and constraint satisfaction properties were obtained. Moreover, the non convexity of the functional used in the "learning phase" of the neural network gives rise to possible deteriorations in the approximation, due to trapping in local minima.

In Canale and Milanese (2005), a Set Membership (SM) approximation technique has been proposed in order to overcome such drawbacks. However, in both Parisini and Zoppoli (1995) and Canale and Milanese (2005) no analysis has been carried out on the effects of the approximated control law on the performance of the closed loop system. Some results in this direction can be found in Johansen and Grancharova (2003) and Bemporad and Filippi (2006), where the use of PWA approximations of f^0 is investigated, using a state space partition in polyhedral regions whose vertices are given by ν exact control moves computed off-line. Feasibility of the approximated control law is shown and performance degradation can be estimated in terms of a conservative bound on the difference between the optimal cost function and the one obtained with $\widehat{f}$. Guaranteed stability and regulation properties depend on this bound, which can be reduced to a desired level of accuracy by increasing the number of partitions. However, all the obtained properties rely on the assumption of convexity of the optimal cost function with respect to the state variables, which may limit the applicability of such approaches. In fact, as discussed in Johansen (2004), when such assumption is not met no systematic procedures can be effectively used to guarantee the same properties of the convex case. Finally, the number of partitions may grow significantly with the desired level of accuracy and with the system dimension, preventing the on-line implementation with small sampling periods.

In this paper, it is shown how SM methodologies (see e.g. Milanese and Novara, 2004) allow to systematically approximate MPC controllers for linear systems. Properties in terms of guaranteed approximation error, closed-loop stability and performance will be derived without any convexity assumption on the optimal cost function, but requiring the continuity of f^0 only. In particular, two SM approximations are proposed. The first one, indicated as f^{OPT}, minimizes the guaranteed approximation error for a given ν, but its on-line computational time increases with ν. The second one, indicated as f^{NP}, has higher guaranteed approximation error, but lower on-line computational time which may be rendered constant with ν.

The effectiveness of the proposed techniques is tested on two examples. The first one is related to a double integrator, where the computational time savings with respect to the explicit MPC solution are shown. The second is a multivariable system with a nonlinear constraint, for which no explicit solution is known and the convexity assumption of the PWA approximation techniques is not met.

The paper is organized as follows. In Section 2 the considered MPC problem is introduced together with prior assumptions on the nominal control law. Section 3 contains the main results regarding stability properties and performance of the approximated controller. Functions f^{OPT} and f^{NP} are introduced in Section 4, together with their approximation properties. Section 5 introduces the numerical examples. Finally, conclusions and further lines of development are reported in Section 6.

2 Model Predictive Control

Consider the following linear state space model:

$$x_{t+1} = Ax_t + Bu_t \tag{1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system state and input respectively. Assume that the problem is to regulate the system state to the origin under some input and state constraints. By defining the prediction horizon N_p and the control horizon $N_c \leq N_p$ (for simplicity the assumption $N_c = N_p = N$ will be adopted), it is possible to define a cost function $J(U, x_{t|t}, N)$ of the form

$$J(U, x_{t|t}, N) = \sum_{k=0}^{N-1} L(x_{t+k|t}, u_{t+k|t}) + F(x_{k+N|k})$$

which is evaluated on the basis of the predicted state values $x_{t+k|t}$, $k = 1, \dots, N$, obtained using the model (1), an input sequence $U = \left[u_{t|t}, \dots, u_{t+N-1|t}\right]$ and the "initial" state $x_{t|t} = x_t$. The per-stage cost function $L(\cdot)$ and the terminal state cost $F(\cdot)$ are chosen according to the desired performances and are continuous in their arguments (see Goodwin *et al.*, 2005). The MPC control law is then obtained according to the receding horizon principle (see e.g. Mayne *et al.*, 2000; Goodwin *et al.*, 2005) based on the following optimization problem:

$$\min_{U} \quad J(U, x_{t|t}, N) \tag{2a}$$

subject to

$$\begin{aligned} &x_{t+k|t} \in \mathbb{X},\ k = 1, \dots, N \\ &u_{t+k|t} \in \mathbb{U},\ k = 0, \dots, N \end{aligned} \tag{2b}$$

Where the input and state constraints are represented by a set $\mathbb{X} \subseteq \mathbb{R}^n$ and a compact set $\mathbb{U} \subseteq \mathbb{R}^m$, both containing the origin in their interiors. As a matter of fact, additional constraints (e.g. state contraction, terminal set, etc.. . .) may be employed in order to ensure stability of the controlled system. It is assumed that the optimization problem (2) is feasible over a set $\mathcal{F} \subseteq \mathbb{R}^n$ which will be referred to as the "feasibility set". The application of the receding horizon procedure gives rise to a nonlinear state feedback control law:

$$u_t = [u_{t,1} \dots u_{t,m}]^T = [f_1^0(x_t) \dots f_m^0(x_t)]^T = f^0(x_t)$$
$$f^0 : \mathbb{R}^n \to \mathbb{R}^m,\ f_i^0 : \mathbb{R}^n \to \mathbb{R},\ i = 1, \dots, m$$

It is supposed that the control law $u_t = f^0(x_t)$ is such that the resulting autonomous system

$$x_{t+1} = F^0(x_t) = Ax_t + Bf^0(x_t) \tag{3}$$

is uniformly asymptotically stable at the origin for any initial condition $x_0 \in \mathcal{F}$, i.e. it is stable and

$$\begin{aligned} &\forall \epsilon > 0,\ \forall \delta > 0\ \exists T \in \mathbb{N} \text{ s.t.} \\ &\|\phi^0(t+T, x_0)\|_2 < \epsilon,\ \forall t \geq 0,\ \forall x_0 \in \mathcal{F} : \|x_0\|_2 \leq \delta \end{aligned} \tag{4}$$

where $\phi^0(t, x_0) = \underbrace{F^0(F^0(\ldots F^0}_{t \text{ times}}(x_0)\ldots))$ is the solution of (3) at time instant t with initial condition x_0. Note that, according to (2b), for any $x \in \mathcal{F}$ the state and input constraints are always satisfied after the first time step. Thus, the set $\mathcal{F} \cap \mathbb{X}$ is positively invariant with respect to system (3):

$$\phi^0(t, x_0) \in \mathcal{F} \cap \mathbb{X}, \forall x_0 \in \mathcal{F} \cap \mathbb{X}, \forall t \geq 0 \tag{5}$$

Moreover, it is supposed that the function f^0 is continuous over the feasibility set $\mathcal{F}$. Such property depends on the characteristics of the optimization problem (2), see at this regard e.g. the recent work of Spjøtvold *et al.* (2007) and the references therein.

3 Stability and performance analysis

As already pointed out, an approximating function $f^{\text{SM}} \approx f^0$, derived using SM methodologies, will be used instead of f^0 to reduce the on-line computational effort.

3.1 Problem settings

The approximating function f^{SM} is derived using a finite number ν of exact solutions of the optimization problem (2):

$$\tilde{u}^k = [\tilde{u}_1^k, \ldots, \tilde{u}_m^k]^T = f^0(\tilde{x}^k), k = 1, \ldots, \nu$$

computed off-line by considering initial conditions $\tilde{x}^k \in \mathcal{X}_\nu = \{\tilde{x}^k,\ k = 1, \ldots, \nu\} \subseteq \mathcal{X}$, being $\mathcal{X}$ a compact subset of $\mathcal{F}$ where the approximation of f^0 is carried out. The set $\mathcal{X}_\nu$ is supposed to be chosen such that the Hausdorff distance $d_H(\mathcal{X}, \mathcal{X}_\nu)$ between $\mathcal{X}$ and $\mathcal{X}_\nu$ (see e.g. Blagovest (1990)) satisfies

$$\lim_{\nu \to \infty} d_H(\mathcal{X}, \mathcal{X}_\nu) = 0 \tag{6}$$

Note that uniform gridding over $\mathcal{X}$ fulfills condition (6). Moreover, it is assumed that the derived approximating function f^{SM} satisfies the following *key properties*:

i) the input constraints are always satisfied:

$$f^{\text{SM}}(x) \in \mathbb{U},\ \forall x \in \mathcal{X} \tag{7}$$

ii) for a given ν, a bound $\zeta(\nu)$ on the pointwise approximation error can be computed:

$$\|f^0(x) - f^{\text{SM}}(x)\|_2 \leq \zeta(\nu) \in \mathbb{R}^+,\ \forall x \in \mathcal{X} \tag{8}$$

such that:

$$\lim_{\nu \to \infty} \zeta(\nu) = 0 \tag{9}$$

Since $\mathcal{X}$ and the image set $\mathbb{U}$ of f^0 are compact sets, continuity of f^0 implies that its components f_i^0, $i = 1, \ldots, m$ are Lipschitz continuous functions over $\mathcal{X}$, i.e. there exist finite constants γ_i, $i = 1, \ldots, m$ such that:

$$\forall x^1, x^2 \in \mathcal{X},\ \forall i \in [1, m], |f_i^0(x^1) - f_i^0(x^2)| \leq \gamma_i \|x^1 - x^2\|_2 \tag{10}$$

Thus, f^0 is Lipschitz continuous over $\mathcal{X}$, i.e.:

$$\forall x^1, x^2 \in \mathcal{X}, \|f^0(x^1) - f^0(x^2)\|_2 \leq \|\gamma\|_2 \|x^1 - x^2\|_2 \tag{11}$$

where $\gamma = [\gamma_1, \ldots, \gamma_m]^T$. Estimates $\widehat{\gamma}_i, i = 1, \ldots, m$ of γ_i can be derived as follows:

$$\widehat{\gamma}_i = \inf\left(\tilde{\gamma}_i : \tilde{u}_i^h + \tilde{\gamma}_i \|\tilde{x}^h - \tilde{x}^k\|_2 \geq \tilde{u}_i^k,\ \forall k, h = 1, \ldots, \nu\right) \tag{12}$$

The next result proves convergence of $\widehat{\gamma}_i$ to $\gamma_i, i = 1, \ldots, m$.

Theorem 1

$$\lim_{\nu \to \infty} \widehat{\gamma}_i = \gamma_i,\ \forall i = 1, \ldots, m$$

Proof. See the Appendix. □

Due to the Lipschitz property (11) of control law $f^0(x)$, function $F^0(x)$ defined in (3) is Lipschitz continuous too over $\mathcal{X}$ with Lipschitz constant:

$$L_F = \|A\| + \|\gamma\|_2 \|B\| \tag{13}$$

The use of $f^{\text{SM}}(x)$ in place of $f^0(x)$ gives rise to the autonomous system:

$$x_{t+1}^{\text{SM}} = F^{\text{SM}}(x_t^{\text{SM}}) = Ax_t^{\text{SM}} + Bf^{\text{SM}}(x_t^{\text{SM}}) \tag{14}$$

whose state trajectory at time instant t with initial condition x_0 is indicated as $\phi^{\text{SM}}(t, x_0) = \underbrace{F^{\text{SM}}(F^{\text{SM}}(\ldots F^{\text{SM}}}_{t \text{ times}}(x_0)\ldots))$.

It is possible to compute an upper bound on the Euclidean norm of the one-step state trajectory perturbation induced by the use of control function f^{SM} instead of f^0. Considering any initial state $x_t \in \mathcal{X}$, such perturbation is computed as:

$$\begin{aligned} x_{t+1}^{\text{SM}} - x_{t+1} &= A(x_t - x_t) + B(f^{\text{SM}}(x_t) - f^0(x_t)) \\ &= B(f^{\text{SM}}(x_t) - f^0(x_t)) = e(x_t) \end{aligned} \tag{15}$$

Therefore, the following state equation is obtained:

$$x_{t+1}^{\text{SM}} = F^0(x_t^{\text{SM}}) + e(x_t^{\text{SM}}) \tag{16}$$

Since $f^0(x)$ is not known in general, $e(x)$ cannot be explicitly computed, but a bound on its Euclidean norm $\forall x \in \mathcal{X}$ can be derived from (8) and (15):

$$\begin{aligned} &\|e(x)\|_2 = \|B(f^{\text{SM}}(x) - f^0(x))\|_2 \\ &\leq \|B\|\,\|f^{\text{SM}}(x) - f^0(x)\|_2 \leq \|B\|\,\zeta(\nu) = \mu(\nu) \end{aligned} \tag{17}$$

The value of $\mu(\nu)$ depends on the number ν of exact solutions of (2) considered for the approximation f^{SM} of f^0. If f^{SM} has the property (9), from (17) it follows that:

$$\lim_{\nu \to \infty} \mu(\nu) = 0 \tag{18}$$

Thus it is always possible to choose a suitable value of ν which guarantees a given one-step perturbation upper bound $\mu(\nu)$.

3.2 Main results

Define the following candidate Lyapunov function $V : \mathcal{X} \to \mathbb{R}^+$ for system (3):

$$V(x) = \sum_{j=0}^{\widehat{T}-1} \|\phi^0(j,x)\|_2 \;\forall x \in \mathcal{X} \tag{19}$$

where $\widehat{T} \geq T$ and $T = \inf\limits_{x \in \mathcal{X}} (T \in \mathbb{N} : \|\phi^0(t+T,x)\|_2 < \|x\|_2,\ \forall t \geq 0)$. The following inequalities hold:

$$\|x\|_2 \leq V(x) \leq \sup_{x\in\mathcal{X}} \frac{V(x)}{\|x\|_2}\|x\|_2 = b\,\|x\|_2,\ \forall x \in \mathcal{X} \tag{20}$$

and

$$\begin{aligned} &V(F^0(x)) - V(x) = \Delta V(x) = \\ &= -\frac{\|x\|_2 - \|\phi^0(\widehat{T},x)\|_2}{\|x\|_2}\|x\|_2 \leq -K\|x\|_2,\ \forall x \in \mathcal{X} \end{aligned} \tag{21}$$

with $K = \inf\limits_{x\in\mathcal{X}} \dfrac{\|x\|_2 - \|\phi^0(\widehat{T},x)\|_2}{\|x\|_2}$, $0 < K < 1$. Thus $V(x)$ is a Lyapunov function for system (3) over $\mathcal{X}$. Moreover, $V(x)$ is Lipschitz continuous with constant

$$\tilde{L}_V = \sum_{j=0}^{\widehat{T}-1} (L_F)^j \tag{22}$$

thus the following inequality holds:

$$\begin{aligned} &\forall x \in \mathcal{X},\ \forall e : (F^0(x) + e) \in \mathcal{X} \\ &V(F^0(x) + e) \leq V(F^0(x)) + \tilde{L}_V \mu \end{aligned} \tag{23}$$

Note that the constant $\tilde{L}_V$ as defined in (22) is not in general the lowest Lipschitz constant for $V(x)$. A less conservative estimate $\hat{L}_V$ of the actual constant L_V can be computed as:

$$\begin{aligned} &\widehat{L}_V = \inf(\tilde{L}_V : V(\tilde{x}^h) + \tilde{L}_V\|\tilde{x}^h - x^k\| \geq V(x^k), \\ &\ \forall x^k, x^h \in \mathcal{X}_\nu) \end{aligned} \tag{24}$$

Similarly to Theorem 1, it can be shown that $\lim\limits_{\nu\to\infty} \widehat{L}_V = L_V$. In the sequel, the following notations will be used:

$$\begin{aligned} &\mathbb{B}(x,r) = \{\widehat{x} \in \mathbb{R}^n \;:\; \|\widehat{x} - x\|_2 \leq r,\ \} \\ &\mathbb{B}(\mathcal{A},r) = \bigcup_{x\in\mathcal{A}} \mathbb{B}(x,r),\ \mathcal{A} \subseteq \mathbb{R}^n \end{aligned}$$

Theorem 2 *Let f^{SM} be an approximation of the exact nonlinear MPC controller f^0, defined over the compact set $\mathcal{X}$, satisfying properties (7), (8) and (9), computed using a number ν of exact off-line solutions. Let $\mathcal{G} \subset \mathcal{X}$ be a positively invariant set with respect to (3), i.e.:*

$$\mathcal{G} \subset \mathcal{X} : \phi^0(t,x_0) \in \mathcal{G},\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0 \tag{25}$$

Then, it is always possible to explicitly compute a suitable finite value of ν such that there exist a finite value $\Delta \in \mathbb{R}^+$ with the following properties:

i) *the distance $d(t,x_0) = \|\phi^{SM}(t,x_0) - \phi^0(t,x_0)\|$ is bounded by Δ:*

$$d(t,x_0) \leq \Delta,\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0 \tag{26}$$

ii) *Δ can be explicitly computed as:*

$$\Delta = \sup_{t\geq 0} \min(\Delta_1(t,\mu), \Delta_2(t,\mu)) \tag{27}$$

where:

$$\Delta_1(t,\mu) = \sum_{k=0}^{t-1} (L_F)^k \mu \tag{28}$$

$$\Delta_2(t,\mu) = 2\,\eta^t \sup_{x_0\in\mathcal{G}} V(x_0) + \frac{b}{K}\,L_V\,\mu \tag{29}$$

with $\eta = \left(1 - \dfrac{K}{b}\right)$, $0 < \eta < 1$.

iii) *$\Delta(\nu)$ converges to 0:*

$$\lim_{\nu\to\infty} \Delta(\nu) = 0 \tag{30}$$

iv) *the state trajectory of system (14) is kept inside the set $\mathbb{B}(\mathcal{G},\Delta)$ for any $x_0 \in \mathcal{G}$:*

$$\phi^{SM}(t,x_0) \in \mathbb{B}(\mathcal{G},\Delta),\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0 \tag{31}$$

v) *the set $\mathbb{B}(\mathcal{G},\Delta)$ is contained in $\mathcal{X}$*

$$\mathbb{B}(\mathcal{G},\Delta) \subseteq \mathcal{X}$$

vi) *the state trajectories of system (14) asymptotically converge to the set $\mathbb{B}(0,q)$, i.e.:*

$$\lim_{t\to\infty} \|\phi^{SM}(t,x_0)\|_2 \leq q,\ \forall x_0 \in \mathcal{G}$$

with

$$q = \frac{b}{K} L_V\, \mu \leq \Delta \tag{32}$$

Proof. See the Appendix. □

Remark 1 *Note that the set $\mathcal{X}$ has to be chosen such that it contains in its interior a set $\mathcal{G}$ satisfying (25). Due to property (5), if the state constraint set $\mathbb{X}$ is bounded and the feasibility set $\mathcal{F}$ is such that $\mathbb{X} \subset \mathcal{F}$, any set $\mathcal{G}$ such that $\mathbb{X} \subseteq \mathcal{G} \subset \mathcal{F}$ is positively invariant with respect to system (3). Moreover, note that $\{0\} \in \mathcal{G}$, since the origin is a stable fixed point for the nominal system (3).*

Remark 2 *If $L_F < 1$ (i.e. F^0 is a contraction operator), a simplified formulation for bound Δ is obtained. In fact, Lyapunov function (19) can be chosen as $V(x) = \|x\|_2$, with $b = 1$ in (20) and $K = (1 - L_F)$ in (21), leading to $L_V = 1$. Thus the bound $\Delta_2(t,\mu)$ in (29) is computed as:*

$$\Delta_2(t,\mu) = 2(L_F)^t \sup_{x_0\in\mathcal{G}} \|x_0\|_2 + \frac{1}{1-L_F}\mu$$

and q in (32) is $q = \dfrac{1}{1-L_F}\,\mu$. On the other hand the bound $\Delta_1(t,\mu)$ in (28) is such that:

$$\Delta_1(t,\mu) \leq \frac{1}{1-L_F}\mu,\ \forall t \geq 0$$

therefore a simpler formulation for Δ is obtained:

$$\Delta = \sup_{t\geq 0} \min(\Delta_1(t,\mu), \Delta_2(t,\mu)) = \frac{1}{1-L_F}\mu$$

Remark 3 *A simplified formulation for bound $\Delta_2(t,\mu)$ is obtained if the MPC problem (2) includes a state contraction constraint (see e.g. Polak and Yang (1993)):*

$$\|\phi^0(t,x_0)\|_2 \leq \sigma\, \|\phi^0(t-1,x_0)\|_2,\ 0 < \sigma < 1$$

In this case, Lyapunov function (19) can be chosen as $V(x) = \|x\|_2$, with $b = 1$ in (20) and $K = (1-\sigma)$ in (21), leading to $L_V = 1$. Thus the bound $\Delta_2(t,\mu)$ in (29) is computed as:

$$\Delta_2(t,\mu) = 2\sigma^t \sup_{x_0\in\mathcal{G}} \|x_0\|_2 + \frac{1}{1-\sigma}\mu$$

and q in (32) is $q = \dfrac{1}{1-\sigma}\,\mu$.

The main consequence of Theorem 2 is that, with the proper value of ν, for any initial condition $x_0 \in \mathcal{G}$ it is guaranteed that the state trajectory is kept inside the set $\mathcal{X}$ and converges to the set $\mathbb{B}(0,q)$, which can be arbitrarily small since q linearly depends on μ, i.e.: $\lim_{\nu\to\infty} q = \left(\frac{b}{K} L_V \lim_{\nu\to\infty} \mu(\nu)\right) = 0$. Moreover, on the basis of (26) and (30) it can be noted that for any $\epsilon > 0$ it is always possible to find a suitable value of ν such that $d(t,x_0) < \epsilon,\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0$. Therefore, for any given required regulation precision $\overline{q}$, using (32) it is possible to compute a priori a sufficient one step perturbation bound $\overline{\mu}$ to guarantee the desired accuracy. Similarly, on the basis of (26)-(29) a bound $\overline{\mu}$ can be computed a priori, such that the trajectory distance is lower than any required maximum value $\overline{\Delta}$. Then, the approximating function f^{SM} can be computed with increasing values of ν, until the corresponding obtained value of μ is such that $\mu \leq \overline{\mu}$, thus guaranteeing the desired performances (i.e. $q \leq \overline{q}$ and/or $\Delta \leq \overline{\Delta}$). Indeed, as $\nu \to \infty$ (i.e. the performances of control system F^{SM} match with those of F^0), the computation time of $f^{\text{SM}}(x)$ increases in general, as well as memory usage. Thus, the value of ν can be chosen in order to set a compromise between system performances, computation times and memory requirements.

Theorem 2 does not address explicitly the problem of state constraint satisfaction for the controlled system (14), i.e.:

$$\phi^{\text{SM}}(t,x) \in \mathbb{X},\ \forall x \in \mathcal{G},\ \forall t \geq 1$$

However, in consequence of Theorem 2, it is possible to choose ν such that there exists a finite number $\overline{T}$ of time steps after which the state trajectory ϕ^{SM} is kept inside the constraint set $\mathbb{X}$, for any initial condition $x_0 \in \mathcal{G}$. Moreover the value of $\overline{T}$ decreases as ν increases. In fact, using (26) it follows that

$$\begin{aligned} &\forall x_0 \in \mathcal{G},\ \forall t \geq 0 \\ &\|\phi^{\text{SM}}(t,x_0)\|_2 \leq \|\phi^0(t,x_0)\|_2 + \Delta(\nu) \end{aligned} \tag{33}$$

Then, considering a value of ν such that:

$$\mathbb{B}(0, \epsilon + \Delta(\nu)) \subset \mathbb{X} \tag{34}$$

with $\epsilon > 0$ "small" enough, on the basis of the uniform asymptotic stability assumption (4), it is always possible to find $\overline{T} < \infty$ such that:

$$\|\phi^0(t+\overline{T},x_0)\|_2 < \epsilon,\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0$$

Using (33) it can be noted that:

$$\begin{aligned} \|\phi^{\text{SM}}(t+\overline{T},x_0)\|_2 \leq \|\phi^0(t+\overline{T},x_0)\|_2 + \Delta(\nu) < \\ < \epsilon + \Delta(\nu),\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0 \\ \Rightarrow \phi^{\text{SM}}(t+\overline{T},x_0) \in \mathbb{B}(0,\epsilon+\Delta(\nu)),\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0 \end{aligned}$$

and, on the basis of (34):

$$\phi^{\text{SM}}(t+\overline{T}, x_0) \in \mathbb{X},\ \forall x_0 \in \mathcal{G},\ \forall t \geq 0$$

thus after a finite number $\overline{T}$ of time steps there is the guarantee that state constraints are satisfied. Note that in general the higher is ϵ in (34), the lower is $\overline{T}$. Since the maximum value of ϵ such that (34) holds is higher as $\Delta(\nu)$ decreases, $\overline{T}$ in general decreases as $\Delta(\nu)$ does, i.e. as ν increases.

4 SM approximation of predictive controllers

Two different SM approximation techniques are now proposed, both leading to the computation of approximating functions enjoying the *key properties* (7), (8) and (9), needed for Theorem 2 to hold. For the sake of simplicity, the case of input saturation constraints is considered, i.e.:
$\mathbb{U} = \{u \in \mathbb{R}^m : \underline{u}_i \leq u_i \leq \overline{u}_i,\ i = 1, \ldots, m\}$.

4.1 "Optimal" approximation

Set Membership approximation methodologies can be employed to obtain an "optimal" (OPT) approximation $f^{\text{OPT}} = [f_1^{\text{OPT}}, \ldots, f_m^{\text{OPT}}]^T$ of function f^0 with the desired properties (7), (8) and (9). For each function $f_i^0,\ i \in [1, m]$, the aim is to derive, from the off-line computed values of $\tilde{u}^k$ and $\tilde{x}^k$ and from known properties of f_i^0, an approximation $\widehat{f}_i$ of f_i^0 and a measure of the approximation error, in term of the $L_p(\mathcal{X})$ norm, $p \in [1, \infty]$, defined as $\|f_i\|_p \doteq \left[\int_{\mathcal{X}} |f_i(x)|^p dx\right]^{\frac{1}{p}}$, $p \in [1, \infty)$ and $\|f_i\|_\infty \doteq \text{ess-}\sup_{x\in\mathcal{X}} |f_i(x)|$. In the following, it is implicitly meant that any i is considered and notation "$\forall i : i = 1, \ldots, m$" is omitted for simplicity of reading.
Function $f_i^0(x)$ is Lipschitz continuous over $\mathcal{X}$, therefore $f_i^0 \in \mathcal{A}_{\gamma_i}$, where $\mathcal{A}_{\gamma_i}$ is the set of all continuous Lipschitz functions f_i on $\mathcal{X}$, with constants γ_i, such that $\underline{u}_i \leq f_i(x) \leq \overline{u}_i,\ \forall x \in \mathcal{X}$. This prior information on function f_i^0, combined with the knowledge of the values of the function at the points $\tilde{x}^k \in \mathcal{X},\ \ k = 1, \ldots, \nu$, lead to:

$$f_i^0 \in FFS_i = \{f_i \in \mathcal{A}_{\gamma_i} : f_i(\tilde{x}^k) = \tilde{u}_i^k,\ \ k = 1, \ldots, \nu\} \tag{35}$$

where FFS_i is named Feasible Functions Set. The aim is to derive an approximation of f_i^0 using the information (35). For given $\widehat{f}_i \approx f_i^0$, the related L_p approximation error is $\|f_i^0 - \hat{f}_i\|_p$. This error cannot be exactly computed, but its tightest bound is given by:

$$\|f_i^0 - \hat{f}_i\|_p \leq \sup_{\tilde{f}_i \in FSS} \|\tilde{f}_i - \hat{f}_i\|_p \doteq E(\widehat{f}_i) \tag{36}$$

where $E(\widehat{f}_i)$ is called (guaranteed) *approximation error*.
A function f_i^{OPT} is called an *optimal approximation* if:

$$E(f_i^{\text{OPT}}) = \inf_{\widehat{f}_i} E(\widehat{f}_i) \doteq r_{p,i} \tag{37}$$

The quantity $r_{p,i}$, called *radius of information*, gives the minimal L_p approximation error that can be guaranteed.
It is also of interest to evaluate, for given $x \in \mathcal{X}$, the tightest lower and upper bounds on $f_i^0(x)$. They are given as:

$$\underline{f}_i(x) \leq f_i^0(x) \leq \overline{f}_i(x), \forall x \in \mathcal{X}$$

where:

$$\overline{f}_i(x) = \sup_{\tilde{f}_i \in FFS_i} \tilde{f}_i(x)$$
$$\underline{f}_i(x) = \inf_{\tilde{f}_i \in FFS_i} \tilde{f}_i(x)$$

are called *optimal bounds*.
The next result gives the solution to the problem of optimal bounds evaluation.

Theorem 3 *The optimal bounds can be computed as:*

$$\begin{aligned} \overline{f}_i &\doteq \min\left[\overline{u}_i, \min_{k=1,\ldots,\nu}\left(\tilde{u}_i^k + \gamma_i\|x - \tilde{x}^k\|_2\right)\right] \in FFS_i \\ \underline{f}_i &\doteq \max\left[\underline{u}_i, \max_{k=1,\ldots,\nu}\left(\tilde{u}_i^k - \gamma_i\|x - \tilde{x}^k\|_2\right)\right] \in FFS_i \end{aligned} \tag{38}$$

Proof. Trivial extension of Theorem 2 in Milanese and Novara (2004), for the case of Lipschitz continuous functions in presence of saturation. □

Finding optimal bounds is also instrumental to solve the optimal approximation problem, as given in the next result.

Theorem 4 ***i)*** *The function:*

$$f_i^{OPT}(x) = \tfrac{1}{2}[\overline{f}_i(x) + \underline{f}_i(x)] \in FFS_i \tag{39}$$

is an optimal approximation for any $L_p(\mathcal{X})$ norm, with $p \in [1, \infty]$

ii) *The radius of information is given by:*

$$r_{p,i} = \frac{1}{2}\|\overline{f}_i - \underline{f}_i\|_p,\ \forall p \in [1, \infty] \tag{40}$$

iii) *For given ν, it results:*

$$\|f_i^0 - f_i^{OPT}\|_p \leq r_{p,i},\ \forall p \in [1, \infty] \tag{41}$$

iv) *The radius of information $r_{\infty,i}$ is bounded:*

$$r_{\infty,i} \leq \gamma_i d_H(\mathcal{X}, \mathcal{X}_\nu) \tag{42}$$

v) *The radius of information $r_{p,i}$ is convergent to zero:*

$$\lim_{\nu\to\infty} r_{p,i} = 0,\ \forall p \in [1, \infty] \tag{43}$$

vi) *The approximation error of f_i^{OPT} is pointwise convergent to zero:*

$$\lim_{\nu\to\infty} |f_i^0(x) - f_i^{OPT}(x)| = 0,\ \forall x \in \mathcal{X}$$

Proof. Trivial extension of Theorem 7 in Milanese and Novara (2004), for the case of Lipschitz continuous in presence of saturation. □

Let us define the function $f^{\text{OPT}} = [f_1^{\text{OPT}}, \ldots, f_m^{\text{OPT}}]^T$. On the basis of (41) it can be noted that:

$$\|f^0(x) - f^{\text{OPT}}(x)\|_2 \le \|r_\infty\|_2, \, \forall x \in \mathcal{X} \qquad (44)$$

with $r_\infty = [r_{\infty,1}, \ldots, r_{\infty,m}]$. Note that, as a consequence of (42), the Euclidean norm of the vector r_∞ is bounded:

$$\|r_\infty\|_2 \le \|\gamma\|_2 \, d_H(\mathcal{X}, \mathcal{X}_\nu) \qquad (45)$$

and, due to (6), it converges to zero as ν increases:

$$\lim_{\nu \to \infty} \|r_\infty\|_2 = 0 \qquad (46)$$

Thus, according to (44) and (46), function f^{OPT} satisfies conditions (8) and (9) with $\zeta^{\text{OPT}} = \|r_\infty\|_2$. The corresponding value of μ in (17) is

$$\mu^{\text{OPT}} = \|B\| \, \zeta^{\text{OPT}} = \|B\| \, \|r_\infty\|_2 \qquad (47)$$

Moreover, since $f_i^{\text{OPT}} \in FFS_i$ (see (39)), input constraints are satisfied and f^{OPT} fullfills condition (7), thus meeting all the properties needed for the stability results of Section 3.

Remark 4 *The closed loop system $F^{OPT}(x) = A(x) + Bf^{OPT}(x)$ results to be Lipschitz continuous with Lipschitz constant L_F. Then if $L_F < 1$, system F^{OPT} results to be a contraction operator and its stability analysis is straightforward, since it is known that exponential asymptotic stability in the origin is guaranteed for such systems (see e.g. Wang* et al. *(2001)).*

4.2 "Nearest Point" approximation

On-line computation times of the function f^{OPT} may result too high for the considered application. In this section, other kinds of approximating functions which satisfy conditions (7)–(9) are sought-after, whose approximation error is not the optimal one, but whose computation is simpler. In particular, a very simple example of such approximating techniques is investigated, denoted as "Nearest Point" (NP) approximation. For a given value of ν, the NP approximation leads in general to an higher approximation error bound ζ^{NP} than OPT approximation, but to lower on-line computation times, whose growth as a function of ν is much slower than that of OPT approximation (see *Example 1* in Section 5). Thus, the NP approximation required to guarantee given stability and performance properties may need much lower on-line computation times with respect to OPT approximation, at the expenses of longer off-line computation time.
For any $x \in \mathcal{X}$, denote with $\tilde{x}^{\text{NP}}$ a state value such that:

$$\tilde{x}^{\text{NP}} \in \mathcal{X}_\nu, \, \|\tilde{x}^{\text{NP}} - x\|_2 = \min_{\tilde{x} \in \mathcal{X}_\nu} \|\tilde{x} - x\|_2 \qquad (48)$$

Then, the NP approximation $f^{\text{NP}}(x)$ is computed as:

$$f^{\text{NP}}(x) = f^0(\tilde{x}^{\text{NP}}) = \tilde{u}^{\text{NP}} \qquad (49)$$

Such approximation trivially satisfies condition (7). The next Theorem 5 shows that NP approximation (49) satisfies also properties (8) and (9).

Theorem 5 ***i)*** *The NP approximation error $\|f^0(x) - f^{NP}(x)\|_2$ is bounded:*

$$\|f^0(x) - f^{NP}(x)\|_2 \le \zeta^{NP} = \|\gamma\|_2 \, d_H(\mathcal{X}, \mathcal{X}_\nu), \; \forall x \in \mathcal{X}$$

ii) *The bound ζ^{NP} converges to zero:*

$$\lim_{\nu \to \infty} \zeta^{NP} = 0$$

Proof. Trivial application of the Lipschitz continuity property of f^0, of the definition of Hausdorff distance $d_H(\mathcal{X}, \mathcal{X}_\nu)$ and of property (6). □

Remark 5 *The NP approximation (49) satisfies the properties (7), (8) and (9) considered in Section 3, with $\zeta^{NP} = \|\gamma\|_2 \, d_H(\mathcal{X}, \mathcal{X}_\nu)$. The corresponding value of μ in (17) is*

$$\mu^{NP} = \|B\| \, \zeta^{NP} = \|B\| \, \|\gamma\|_2 \, d_H(\mathcal{X}, \mathcal{X}_\nu) \qquad (50)$$

Since $\|r_\infty\|_2 \le \|\gamma\|_2 \, d_H(\mathcal{X}, \mathcal{X}_\nu)$ (see (45)), it can be noted that:

$$\mu^{\text{OPT}}(\nu) = \|B\| \|r_\infty\|_2 \le \|B\| \, \|\gamma\|_2 \, d_H(\mathcal{X}, \mathcal{X}_\nu) = \mu^{\text{NP}}$$

Thus in general the one step perturbation bound obtained using OPT approximation is lower than the one obtained with NP approximation. However, with NP approximation it is possible to obtain the same value of the one step perturbation bound $\mu^{\text{OPT}}(\nu)$ using a higher number of off-line evaluations of the MPC control law, i.e. there exist a finite value $\nu' > \nu$ such that: $\mu^{\text{NP}}(\nu') \le \mu^{\text{OPT}}(\nu)$. Due to the simplicity of f^{NP}, the on-line computational times needed to evaluate the NP approximation based on ν' off-line computed values may be much lower than the one needed to evaluate the OPT approximation based on ν off-line computations.

5 Simulation examples

5.1 Example 1: double integrator

Consider the double integrator system:

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_t$$

A predictive controller is designed using a quadratic cost function J:

$$J(U, x_{t|t}, N) = x_{t+N|t}^T P x_{t+N|t} + \sum_{k=0}^{N-1} \{x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^T R u_{t+k|t}\} \tag{51}$$

where $P \succ 0$, $Q = Q^T \succ 0$ and $R = R^T \succ 0$ are positive definite matrices. The following choice has been made in the considered example:

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix},\ R = 1,\ P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ N = 5$$

Input and output constraints are defined by:

$$\mathbb{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\},\ \mathbb{U} = \{u \in \mathbb{R} : |u| \leq 1\}$$

The MATLAB® Multi-Parametric Toolbox (Kvasnica *et al.*, 2006) has been used to compute the explicit MPC solution. The obtained feasibility set $\mathcal{F}$ is reported in Fig. 1. The number of regions (after the merging of regions with the same control law) over which the nominal control law f^0 is affine is equal to 5. The computed values of the Lipschitz constants (12) and (13) are $\gamma = 1.4$ and $L_F = 3.19$ respectively. The set $\mathcal{X} = \mathcal{F}$ has been considered for the approximation of f^0 and Lyapunov function (19) has been computed with $\widehat{T} = 7$: the resulting values of b and K in (20) and (21) are $b = 3.15$, $K = 0.99$, while $\widehat{L}_V$ of (24) is $\widehat{L}_V = 8.1$.

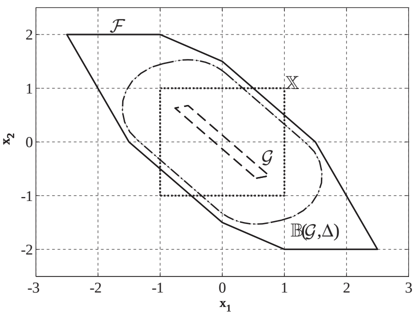

Fig. 1. Example 1: sets $\mathcal{F} = \mathcal{X}$ (solid line), $\mathcal{G}$ (dashed line), $\mathbb{B}(\mathcal{G}, \Delta)$ (dash-dotted line) and $\mathbb{X}$ (dotted line). Sets $\mathcal{G}$ and $\mathbb{B}(\mathcal{G}, \Delta)$ obtained using OPT approximation with $\nu \simeq 1.6\ 10^6$.

Assume that the required regulation precision is $\|x_t\|_2 \leq \overline{q} = 5\,10^{-2}$ for $t \to \infty$. According to (32), the corresponding sufficient value of μ is equal to $\overline{\mu} = (\overline{q}\,K)/(b\,L_V) = 1.9\,10^{-3}$. By performing OPT approximation f^{OPT} of f^0 with $\nu \simeq 1.6\ 10^6$, a value of $\mu = 1.4\,10^{-3} < \overline{\mu}$ is obtained, which leads to $q = 3.7\ 10^{-2} < \overline{q}$. The corresponding upper bound Δ (26) on distance trajectories can be computed using (27), via the computation of the bounds $\Delta_1(t)$ (28) and $\Delta_2(t)$ (29): the obtained value is $\Delta = 0.849$. The obtained set $\mathcal{G}$ and the corresponding set $\mathbb{B}(\mathcal{G}, \Delta) \subseteq \mathcal{F}$ (31) are reported in Fig. 1. As a matter of fact, the obtained properties of the system regulated using the approximated controller are quite good despite the computed theoretical values of Δ and q. This fact highlights that the stability and performance conditions claimed in Theorem 2 may prove to be conservative, being only sufficient. Indeed, with a much lower number ν of off-line solutions, stability and performance are kept for any $x_0 \in \mathcal{X}$. Clearly, a lower number of off-line solutions leads to lower computational efforts and memory usage: to evaluate the on-line computational times as well as performance degradation obtained with the approximated control law, a number N^{SIM} of simulations have been performed, considering any initial condition x_0^{SIM} computed via uniform gridding over $\mathcal{X}$ with a resolution equal to 0.01 for both state variables. Each simulation lasted 500 time steps. Then, the mean computational time $\overline{t}$ over all the initial conditions and all the time steps of each simulation has been computed, together with the maximum trajectory distance obtained over all the simulations:

$$\Delta^{\text{SIM}} = \max_{x_0^{\text{SIM}}} \left(\max_{t \in [1,500]} \left(\|\phi^{\text{SM}}(t, x_0^{\text{SIM}}) - \phi^0(t, x_0^{\text{SIM}})\|_2 \right) \right)$$

The following estimate of regulation precision has been also considered:

$$q^{\text{SIM}} = \max_{x_0^{\text{SIM}}} \left(\max_{t \in [301,500]} \left(\|\phi^{\text{SM}}(t, x_0^{\text{SIM}})\|_2 \right) \right)$$

Finally, also the mean value $\overline{\Delta}_u$ and the maximum value Δ_u^{MAX} of the approximation error $\|f^0(x) - f^{\text{SM}}(x)\|_2$ over all time instants of all simulations have been considered. These values have been computed employing different values of ν: the obtained results in the case of OPT approximation are reported in Table 1, together with the theoretical values $\Delta(\nu)$, $q(\nu)$ and $\zeta(\nu)$ obtained using the results of Theorems 2 and 4. As it was expected, the obtained estimates of the maximum trajectory distance Δ^{SIM}, regulation precision q^{SIM} and mean and maximum approximation errors $\overline{\Delta}_u$ and Δ_u^{MAX} are bounded by their respective theoretical values, Δ, q and ζ. However, these bounds are not strict, being obtained on the basis of sufficient conditions only. Moreover, note that with any considered value of ν the state trajectory has been always kept inside the set $\mathcal{X}$ for any considered initial condition and inside the constraint set $\mathbb{X}$ for any $t \geq 1$. Finally, variable u always satisfied the input constraints, as it was expected. The obtained computational times depend on the employed calculator and on the algorithm implementation: in this case MATLAB® 7 and an AMD Athlon(tm) 64 3200+ with 1 GB RAM have been used and no particular effort was made to optimize the numerical computation of $f^{\text{OPT}}(x)$. On the same platform, the mean computational time obtained with on-line optimization (using the MATLAB® `quadprog` function) is about $2.5\,10^{-2}$ s, while the mean computational time obtained with the toolbox developed by Kvasnica *et al.* (2006) for the calculation of the explicit solution is about $2.2\,10^{-3}$ s. In this example, NP approximation has been tested too, using the same off-line computed values of $f^0(\tilde{x}^k)$ employed for the OPT approximation. Table 2 contains the estimates of mean

Table 1
Example 1: properties of approximated MPC using OPT approximation.

	$\nu \simeq 1.6\ 10^6$	$\nu \simeq 10^5$	$\nu \simeq 5\ 10^3$	$\nu \simeq 10^3$
$\bar{t}$	$5.4\,10^{-1}$ s	$2.2\,10^{-2}$ s	$7.8\,10^{-4}$ s	$3.8\,10^{-4}$ s
Δ^{SIM}	$1.6\ 10^{-9}$	$1\,10^{-2}$	$3\,10^{-2}$	$9\,10^{-2}$
Δ	$8.5\ 10^{-1}$	1.35	2.5	3.2
q^{SIM}	$1.7\ 10^{-16}$	$4\,10^{-9}$	$4\,10^{-6}$	$1.5\,10^{-4}$
q	$3.7\,10^{-2}$	$1.6\,10^{-1}$	$7.8\,10^{-1}$	1.5
$\overline{\Delta}_u$	$2.4\ 10^{-12}$	$5.9\,10^{-11}$	$4.3\,10^{-7}$	$2.5\,10^{-3}$
Δ_u^{MAX}	$4.5\ 10^{-11}$	$7.4\,10^{-10}$	$8.8\,10^{-6}$	$1\,10^{-2}$
ζ	$1.3\,10^{-3}$	$5.5\,10^{-3}$	$2.7\,10^{-2}$	$5.2\,10^{-2}$

computational time, maximum trajectory distance, regulation precision and approximation errors obtained with NP approximation and different values of ν, together with the theoretical values $\Delta(\nu)$, $q(\nu)$ and $\zeta(\nu)$. Note that the evaluation times of OPT approximation grow linearly with ν, while those obtained with NP approximation are practically constant: this can be obtained with a suitable storage criterion for the off-line computed data, which leads to computational times that depend on the number of state variables but not on the value of ν. In all the performed simulations, uniform gridding over $\mathcal{X}$ has been used to obtain the set $\mathcal{X}_\nu$ and to compute the corresponding exact control moves $\tilde{u}^k, k = 1, \ldots, \nu$. In order to improve the regulation precision of both OPT and NP approximated control laws, it is also possible to employ a more dense gridding of exact MPC solutions near the origin.

Table 2
Example 1: properties of approximated MPC using NP approximation.

	$\nu \simeq 1.6\ 10^6$	$\nu \simeq 10^5$	$\nu \simeq 5\ 10^3$	$\nu \simeq 10^3$
$\bar{t}$	$3.5\,10^{-5}$ s	$4\,10^{-5}$ s	$4.5\,10^{-5}$ s	$2.6\,10^{-5}$ s
Δ^{SIM}	$3.4\,10^{-3}$	$1.5\,10^{-2}$	$6.5\,10^{-2}$	$1.3\,10^{-1}$
Δ	1.3	2	3.9	5.4
q^{SIM}	$3.2\,10^{-3}$	$1.3\,10^{-2}$	$4.7\,10^{-2}$	$1.3\,10^{-1}$
q	$7.1\,10^{-2}$	$2.8\,10^{-1}$	1.4	2.9
$\overline{\Delta}_u$	$4.7\ 10^{-4}$	$1.7\,10^{-3}$	$2\,10^{-2}$	$5\,10^{-2}$
Δ_u^{MAX}	$1.3\ 10^{-3}$	$3\,10^{-3}$	$3\,10^{-2}$	$7\,10^{-2}$
ζ	$2.6\,10^{-3}$	$5\,10^{-3}$	$5\,10^{-2}$	$1\,10^{-1}$

5.2 *Example 2: two inputs system with state contraction constraint*

In this example, the following two inputs system, originally introduced in Zheng *et al.* (1994), is considered:

$$x_{t+1} = \begin{bmatrix} 0.98 & 0 \\ 0 & 0.98 \end{bmatrix} x_t + \begin{bmatrix} 0.8 & -1 \\ -0.6 & 0.8 \end{bmatrix} u_t$$

State and input constraints are also taken into account:

$$\mathbb{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 2\},\ \mathbb{U} = \{u \in \mathbb{R}^2 : \|u\|_\infty \leq 1\}$$

An MPC control law has been designed using a quadratic cost function (51) with the following parameters

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ N = 5$$

Moreover, a state contraction constraint has been added:

$$\|x_{t+1|t}\|_2 \leq \sigma \|x_{t|t}\|_2$$

with $\sigma = 0.96$. The MOSEK© optimization toolbox for MATLAB® (MOSEK ApS (2006)) has been employed to evaluate the Feasibility set $\mathcal{F}$ and to compute off-line the needed values of $f^0(x)$. The set $\mathcal{X} = \mathcal{F}$ considered for the approximation of f^0 is reported in Fig. 2 the level curves of the optimal cost function $J(U^*(x))$. Note that the optimal cost function is not convex, due to the presence of the contraction constraint. Therefore, in this case stability and constraint satisfaction properties cannot be guaranteed with the procedure proposed by Johansen and Grancharova (2003). Moreover, $f^0(x)$ results to be continuous but it is not piece-

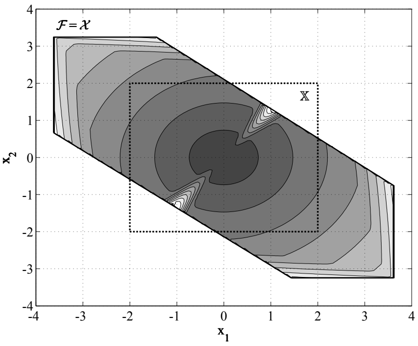

Fig. 2. Example 2: set $\mathcal{F} = \mathcal{X}$ (solid), constraint set $\mathbb{X}$ (dotted) and level curves of the optimal cost function $J(U^*(x))$.

wise affine. In fact, no explicit solution can be easily obtained in this case. The Lipschitz constants γ_1 and γ_2 have been estimated according to (12) as $\gamma_1 = 5.33$, $\gamma_2 = 4.48$. The resulting value of L_F in (13) is $L_F = 12.29$. The Lyapunov function parameters are $b = 1$, $L_V = 1$, $K = 0.04$ (see Remark 2 in Section 3). NP approximation has been carried out employing $\nu \simeq 4.3\,10^5$ exact MPC solutions, obtaining $\Delta = 15.04$ and $q = 1.99$. A comparison of the state courses is shown in Fig. 3, starting from the initial state $x_0 = [-3,\ 0.4]^T$. The approximated control law has the same properties of the nominal one, i.e. state and input constraints are satisfied and the obtained maximum trajectory distance is lower than $7\,10^{-3}$, while the regulation precision is lower than $1\,10^{-3}$. Fig. 4 shows the behaviour of the contraction ratio $\|x_{t+1}\|_2/\|x_t\|_2$: note that the two curves match, thus also the contraction constraint is satisfied

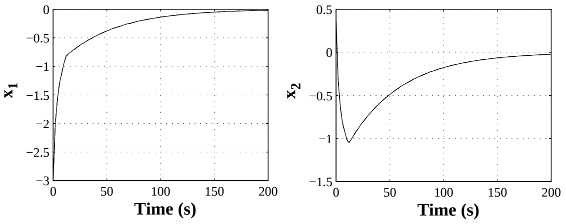

Fig. 3. Example 2: nominal state course (dashed line) and the one obtained with the approximated control law (solid line). Initial state: $x_0 = [-3,\, 0.4]^T$. Approximation carried out with NP approach and $\nu \simeq 4.3\, 10^5$.

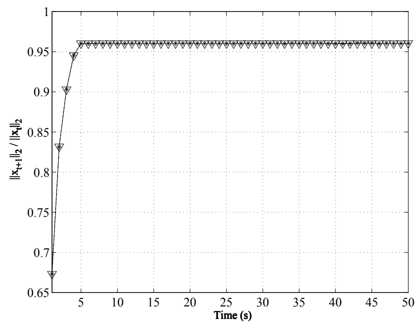

Fig. 4. Example 2: contraction ratio $\|x_{t+1}\|_2/\|x_t\|_2$ of the nominal state trajectory (dashed line with triangles) and of the one obtained with the approximated control law (solid line with asterisks). Initial state: $x_0 = [-3,\, 0.4]^T$. Approximation carried out with NP approach and $\nu \simeq 4.3\, 10^5$.

with the NP approximated control law. As regards the evaluation times, the mean computational time obtained with MOSEK© is equal to 0.016 s, while the NP approximation mean computational time is about $3\ 10^{-5}$ s, thus showing the good computational speed improvement obtained with the approximated controller.

6 Conclusions

The use of SM function approximation methodologies for fast implementation of Model Predictive Control laws for linear systems has been investigated. Conditions on the approximating function have been provided in order to guarantee stability, performance and state constraint satisfaction properties. The only needed assumption is the continuity of the nominal stabilizing MPC solution on the compact set over which the approximation is performed.

Two different SM approximation techniques have been presented, which lead to approximating functions with a desired level of accuracy, fulfilling input constraints and whose computational time is independent on the MPC control horizon. Both methods are based on the off-line computation of a finite number ν of exact MPC control moves. The first method derives an "optimal" approximating function which minimizes, for a given ν, the guaranteed accuracy level. The second one gives lower guaranteed accuracy, but its computational time is lower and it is approximately constant with ν. The choice between the two approximations can be then performed by suitably trading between on-line computational times from one side and off-line computational times and memory requirements from the other. The effectiveness of the proposed methodology has been shown by the application to a double integrator example and to a MIMO plant with a contraction constraint. Note that such approach has also been successfully applied to practical control problems such as semi-active suspension control and energy generation using tethered airfoils (see Canale *et al.* (2006) and Canale *et al.* (2007)).

In conclusion, the SM techniques proposed in this paper appear to be able to cope with the limitations of the other existing approaches for MPC approximation, since they can be systematically applied giving guaranteed performances also to the case of nonlinear constraints and for non-convex optimal cost function. In such cases, an exact explicit piecewise affine MPC solution (or approximation) with guaranteed stability is critical or not possible to compute. Moreover, the SM methodology appears to overcome the problems related to guaranteed approximation error, stability and state constraints satisfaction properties in neural networks approaches.

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A Proofs

Proof of Theorem 1. For any $x^1, x^2 \in \mathcal{X}$, consider two values $\tilde{x}^1, \tilde{x}^2 \in \mathcal{X}_\nu$ such that:

$$\|x^1 - \tilde{x}^1\|_2 \leq d_H(\mathcal{X}, \mathcal{X}_\nu),\ \|x^2 - \tilde{x}^2\|_2 \leq d_H(\mathcal{X}, \mathcal{X}_\nu)$$

Property (6) leads to:

$$\begin{aligned} 0 \leq \lim_{\nu\to\infty} \|x^1 - \tilde{x}^1\|_2 \leq \lim_{\nu\to\infty} d_H(\mathcal{X}, \mathcal{X}_\nu) = 0; \\ 0 \leq \lim_{\nu\to\infty} \|x^2 - \tilde{x}^2\|_2 \leq \lim_{\nu\to\infty} d_H(\mathcal{X}, \mathcal{X}_\nu) = 0; \end{aligned}$$

which implies that

$$\lim_{\nu\to\infty} \tilde{x}^1 = x^1,\ \forall x^1 \in \mathcal{X},\ \lim_{\nu\to\infty} \tilde{x}^2 = x^2,\ \forall x^2 \in \mathcal{X} \qquad (.1)$$

For any $i \in [1, m]$, the estimate $\widehat{\gamma}_i$ (12) of γ_i is such that:

$$\tilde{u}_i^h + \widehat{\gamma}_i \|\tilde{x}^h - \tilde{x}^k\|_2 \geq \tilde{u}_i^k,\ \forall \tilde{x}^h, \tilde{x}^k \in \mathcal{X}_\nu$$

which implies that:

$$\begin{aligned} &\forall \tilde{x}^h, \tilde{x}^k \in \mathcal{X}_\nu, \\ &f_i^0(\tilde{x}^k) - f_i^0(\tilde{x}^h) = \tilde{u}_i^k - \tilde{u}_i^h \leq \widehat{\gamma}_i \|\tilde{x}^h - \tilde{x}^k\|_2 \\ &f_i^0(\tilde{x}^h) - f_i^0(\tilde{x}^k) = \tilde{u}_i^h - \tilde{u}_i^k \leq \widehat{\gamma}_i \|\tilde{x}^h - \tilde{x}^k\|_2 \end{aligned}$$

$$\Rightarrow |f_i^0(\tilde{x}^h) - f_i^0(\tilde{x}^k)| \leq \widehat{\gamma}_i \|\tilde{x}^h - \tilde{x}^k\|_2,\ \forall \tilde{x}^h, \tilde{x}^k \in \mathcal{X}_\nu \qquad (.2)$$

According to (.1), as $\nu \to \infty$ inequality (.2) holds for any $x^1, x^2 \in \mathcal{X}$, therefore $\widehat{\gamma}_i$ tends to satisfy definition (10) and to approximate the Lipschitz constant γ_i of f_i^0 on $\mathcal{X}$ for any $i = 1, \ldots, m$. □

Proof of Theorem 2.

i)–iii) Choose any $x_0 \in \mathcal{G}$ as initial condition for system (14). On the basis of (13), (16) and (17) it can be noted that:

$$\begin{aligned} &d(1, x_0) = \|\phi^{\text{SM}}(1, x_0) - \phi^0(1, x_0)\|_2 = \\ &= \|F^0(x_0) + e(x_0) - F^0(x_0)\|_2 = \|e(x_0)\|_2 \leq \mu \\ &d(2, x_0) = \|\phi^{\text{SM}}(2, x_0) - \phi^0(2, x_0)\|_2 = \\ &= \|F^0(\phi^{\text{SM}}(1, x_0)) + e(\phi^{\text{SM}}(1, x_0)) - F^0(\phi^0(1, x_0))\|_2 \leq \\ &\leq \|e(\phi^{\text{SM}}(1, x_0))\|_2 + \|F^0(\phi^{\text{SM}}(1, x_0)) - F^0(\phi^0(1, x_0))\|_2 \leq \\ &\leq \mu + L_F \|\phi^{\text{SM}}(1, x_0) - \phi^0(1, x_0)\|_2 \leq \mu + L_F \mu \\ &\ldots \\ &d(t, x_0) = \|\phi^{\text{SM}}(t, x_0) - \phi^0(t, x_0)\|_2 \leq \sum_{k=0}^{t-1} (L_F)^k \mu \end{aligned}$$

Thus, the following upper bound of the distance between trajectories $\phi^{\text{SM}}(t, x_0)$ and $\phi^0(t, x_0)$ is obtained:

$$d(t, x_0) \leq \sum_{k=0}^{t-1} (L_F)^k \mu = \Delta_1(t, \mu),\ \forall x_0 \in \mathcal{G},\ \forall t \geq 1 \qquad (.3)$$

As $t \to \infty$ the bound Δ_1 may converge, if $L_F < 1$, or diverge, if $L_F \geq 1$. Assuming that $L_F \geq 1$ (see Remark 2 for the other case), it cannot be proved, on the basis of inequality (.3) alone, that the trajectory distance $d(t, x_0)$ is bounded. On the other hand, by using the properties of Lyapunov function (19) it is possible to compute another upper bound $\Delta_2(t, \mu)$ of $d(t, x_0)$. First of all, through equations (21) and (23) the following inequality is obtained:

$$\begin{aligned} &\forall x \in \mathcal{X},\ \forall e : (F^0(x) + e) \in \mathcal{X} \\ &V(F^0(x) + e) \leq V(x) - K\|x\|_2 + L_V \mu \end{aligned} \qquad (.4)$$

On the basis of (20) and (.4), the state trajectory $\phi^{\text{SM}}(t, x_0)$ is such that:

$$\begin{aligned} &\|\phi^{\text{SM}}(t, x_0)\|_2 \leq V(\phi^{\text{SM}}(t, x_0)) \leq \\ &V(\phi^{\text{SM}}(t-1, x_0)) - K\|\phi^{\text{SM}}(t-1, x_0)\|_2 + L_V \mu \leq \\ &\leq V(\phi^{\text{SM}}(t-1, x_0)) - \frac{K}{b} V(\phi^{\text{SM}}(t-1, x_0)) + L_V \mu \leq \\ &\leq \eta V(\phi^{\text{SM}}(t-1, x_0)) + L_V \mu \leq \\ &\ldots \leq \eta^t V(x_0) + \sum_{j=0}^{t-1} \eta^j L_V \mu \leq \eta^t V(x_0) + \frac{1}{1-\eta} L_V \mu \end{aligned}$$

with $\eta = \left(1 - \dfrac{K}{b}\right) < 1$. Thus, the following result is obtained:

$$\begin{aligned} &\|\phi^{\text{SM}}(t, x_0)\|_2 \leq \eta^t V(x_0) + \frac{b}{K} L_V \mu \\ &\|\phi^0(t, x_0)\|_2 \leq \eta^t V(x_0) \end{aligned} \qquad (.5)$$

Inequalities (.5) can be used to obtain the upper bound $\Delta_2(t,\mu)$ of the distance between nominal and perturbed state trajectories:

$$
\begin{aligned}
&d(t,x_0) = \|\phi^{\mathrm{SM}}(t,x_0) - \phi^0(t,x_0)\|_2 \leq \\
&\leq \|\phi^{\mathrm{SM}}(t,x_0)\|_2 + \|\phi^0(t,x_0)\|_2 \leq 2\,\eta^t V(x_0) + \frac{b}{K}\,L_V\,\mu \leq \\
&\leq 2\,\eta^t \sup_{x_0\in\mathcal{G}} V(x_0) + \frac{b}{K}\,L_V\,\mu = \Delta_2(t,\mu)\,,\ \forall x_0 \in \mathcal{X}\,,\ \forall t \geq 0
\end{aligned}
$$

Note that, since $\mu < \infty$ and $\mathcal{X}$ is compact:

$$
\begin{aligned}
&\Delta_2(t,\mu) < \infty,\ \forall t \geq 0 \\
&\lim_{t\to\infty} \Delta_2(t,\mu) = \frac{b}{K}\,L_V\,\mu = q \\
&q < \Delta_2(t,\mu) < \infty,\ \forall t \geq 0
\end{aligned}
$$

Thus, as t increases towards ∞, the bound $\Delta_2(t,\mu)$ (29) decreases monotonically from a finite positive value equal to $2\sup_{x_0\in\mathcal{G}} V(x_0) + \frac{b}{K}\,L_V\,\mu$ towards a finite positive value q, while the bound $\Delta_1(t,\mu)$ (28) increases monotonically from 0 to ∞. Therefore, for a fixed value of μ there exists a finite discrete time instant $\widehat{t} > 0$ such that $\Delta_1(\widehat{t},\mu) > \Delta_2(\widehat{t},\mu)$. As a consequence, by considering the lowest bound between $\Delta_1(t,\mu)$ and $\Delta_2(t,\mu)$ for any $t \geq 0$, the following bound $\Delta(\mu)$ of $d(t,x)$, which depends only on μ, is obtained:

$$
\Delta(\mu) = \sup_{t\geq 0}\ \min(\Delta_1(t,\mu), \Delta_2(t,\mu))
$$

$$
\begin{gathered}
q \leq \Delta(\mu) < \infty \\
\|\phi^{\mathrm{SM}}(t,x_0) - \phi^0(t,x_0)\|_2 \leq \Delta(\mu),\ \forall x_0 \in \mathcal{G}, \forall t \geq 0
\end{gathered}
$$

Since for any fixed positive value $\tilde{t}$ of t both $\Delta_1(\tilde{t},\mu)$ and $\Delta_2(\tilde{t},\mu)$ increase linearly with $\mu(\nu)$, on the basis of (18) $\Delta(\nu)$ is such that

$$
\lim_{\nu\to\infty} \Delta(\nu) = 0 \tag{.6}
$$

iv)–v) On the basis of (.6), it is possible to tune ν such that, for any initial condition $x_0 \in \mathcal{G} \subset \mathcal{X}$, $\Delta(\mu)$ is as small as needed. Indeed, it is needed that $\phi^{\mathrm{SM}}(t,x_0) \in \mathcal{X}$ for all $t \geq 0$ for all the considered assumptions to hold. Since by hypothesis the set $\mathcal{G}$ (25) is positively invariant for the nominal state trajectories, for a given value of $\Delta(\mu)$ the perturbed state trajectories are such that $\phi^{\mathrm{SM}}(t,x_0) \in \mathbb{B}(\mathcal{G},\Delta(\mu))$, $\forall x_0 \in \mathcal{G}$, $\forall t \geq 0$. Thus, it is sufficient to choose ν such that $\mathbb{B}(\mathcal{G},\Delta(\mu)) \subseteq \mathcal{X}$. Such choice is always possible in the considered context.

vi) On the basis of (.5) and (20) it can be noted that:

$$
\begin{aligned}
\lim_{t\to\infty} \|\phi^{\mathrm{SM}}(t,x_0)\|_2 &\leq \lim_{t\to\infty} \eta^t b\|x_0\|_2 + \frac{b}{K}\,L_V\mu \\
&= \frac{b}{K}\,L_V\mu = q,\ \forall x_0 \in \mathcal{G}
\end{aligned}
$$

□