Direct feedback control design for nonlinear systems

C. Novara [a] L. Fagiano [b] M. Milanese [c]

[a] Dip. di Automatica e Informatica, Politecnico di Torino, Italy. E-mail: carlo.novara@polito.it

[b] Dip. di Automatica e Informatica, Politecnico di Torino, Italy and Dept. of Mechanical Engineering, University of California at Santa Barbara, USA. E-mail: lorenzo.fagiano@polito.it, lorenzo.fagiano@engineering.ucsb.edu

[c] Kitenergy s.r.l. E-mail: mario.milanese@kitenergy.net

Abstract

We propose an approach for the direct design from data of controllers finalized at solving tracking problems for nonlinear systems. This approach, called Direct FeedbacK (DFK) design, overcomes relevant problems typical of the standard design methods, such as modeling errors, non-trivial parameter identification, non-convex optimization, and difficulty in nonlinear control design. Considering a Set Membership (SM) approach, we provide three main contributions. The first one is a theoretical framework for the stability analysis of nonlinear feedback control systems, in which the controller $\widehat{f}$ is an approximation identified from data of an ideal inverse model f_o. In this framework, we derive sufficient conditions under which $\widehat{f}$ stabilizes the closed-loop system. The second contribution is a technique for the direct design of an approximate controller f^* from data, having suitable optimality, stability, and sparsity properties. In particular, we show that f^* is an almost-optimal controller (in a worst-case sense), and we derive a guaranteed accuracy bound, which can be used to quantify the performance level of the DFK control system. We also show that, when the number of data used for control design tends to infinity and these data are dense in the controller domain, the closed-loop stability is guaranteed for a set of trajectories of interest. The technique is based on convex optimization and sparse identification methods, and thus avoids the problem of local minima and allows an efficient on-line controller implementation in real-world applications. The third contribution is a simulation study, regarding the application of DFK to the challenging problem of control design for a class of airborne wind energy generators.

1 Introduction

The reference tracking control problem can be viewed as a dynamics inversion problem. That is, given a reference (or desired) solution $\boldsymbol{r}$ for the (nonlinear) system S to be controlled, find the input $\boldsymbol{u}$ such that the actual system solution $\boldsymbol{x}$ is "close" to $\boldsymbol{r}$.

Most control design techniques rely on a two-step approach to solve this problem: at first, a mathematical model M of the system S is derived; then, a feedback control algorithm $\boldsymbol{u} = f(\boldsymbol{r}, \boldsymbol{x}, \ldots)$ is designed on the basis of M, according to different possible procedures. The inputs to this model-based design approach are the model M and some given performance specifications. The model M has to satisfy two, often contrasting, properties: on the one hand, it has to be simple, in order to allow the design of f, while, on the other hand, it has to describe with sufficient accuracy the dynamical behavior of S in the neighborhood of a given reference $\boldsymbol{r}$, or for a certain set of references of interest. Typically, M is derived on the basis of first-principles equations, which dictate its structure and involve a set of parameters to be identified. We call these kinds of models "physical". In order to achieve the above-mentioned features, the parameters of the physical models are estimated by means of identification procedures, which employ a set of input-output measurements collected from the system S. The described modeling-identification-control design methodology represents the common practice in industrial applications of automatic control and it works nicely in many contexts. However, in several other cases, it is difficult to derive a simple-but-accurate physical model. It may also happen that the identification procedure fails to estimate the model parameters with sufficiently good accuracy. This is typically the case of systems for which there are no simple first-principles laws to be used to derive an accurate model, and/or systems with strong nonlinearities. Another relevant problem of the approaches relying on physical modeling is the design of the control algorithm which, in the case of a nonlinear system, can be very difficult from both the theoretical and applicative points of view.

These situations motivate the research of data-driven inverse control techniques, which are roughly classifiable in two different categories: indirect techniques, and direct ones. Indirect techniques replace the physical model with a "black-box" one, usually in the form of a sufficiently rich combination of basis functions, and use the measured data to identify the model parameters. Then, the controller is derived with different possible approaches that involve some kind of inversion of the identified model. Several approaches based on neural networks follow this indirect procedure for inverse control (see (Levin & Narendra 1996), the survey (Cabrera & Narendra 1999), and the references therein). In principle, indirect techniques rely on the capabilities of neural networks (or other kinds of approximating functions) to approximate any continuous function on a compact set, in order to obtain good approximations of both the direct model and its inverse. However, obtaining stability and performance results for indirect inverse control techniques is quite difficult and, moreover, the use of neural approximations may give rise to issues like trapping in local minima during the training phase. Direct techniques for inverse control aim at avoiding the identification of a model of the system to control, and to obtain the inverse model directly from the available input-output data, hence the name Direct Inverse Control (DIC) (Anuradha, Reddy & Murthy 2009), (Abreu, Teixeira & Ribeiro 2000), (Norgaard, Ravn, Poulsen & Hansen 2000), (Campi & Savaresi 2006). Most of the existing studies concerned with DIC of nonlinear systems also use neural networks to approximate the inverse controller (Anuradha et al. 2009), (Abreu et al. 2000), (Norgaard et al. 2000). Conceptually, the direct design of the control law eliminates the issues related to the uncertainty due to neglected dynamics and parameter estimation errors, which may affect the model-based and indirect data-driven design techniques. However, there is actually no solid theoretical framework regarding the design of DIC laws, and the existing results are justified mainly by means of simulation studies. Moreover, the use of neural networks, or of other kinds of approximating functions whose training involves non-convex optimization, may still cause the above-mentioned issue of local minima.

In this paper, we propose an approach, named Direct FeedbacK (DFK) design, for the direct synthesis from data of control laws for nonlinear state-space systems, allowing to overcome the problems mentioned above. Considering a Set Membership (SM) approach, we provide three main contributions. The first one is to lay out a theoretical framework for the stability analysis of nonlinear feedback control systems, in which the control law $\widehat{f}$ is an approximation of an ideal inverse controller f_o. The latter is supposed to be not known and is approximated from measured data. In this theoretical framework, we derive sufficient conditions under which $\widehat{f}$ stabilizes the closed-loop system for a set of trajectories of interest. The second contribution is to present a technique for the direct design of a controller f^* from data, having suitable optimality, stability, and sparsity properties. In particular, we show that, under closed-loop stability conditions, f^* is an almost-optimal controller (in a worst-case sense), and we derive a guaranteed accuracy bound, which can be used to quantify the performance level of the DFK control system. We also show that, when the number of data used for control design tends to infinity and these data are dense in the controller domain, the controller f^* guarantees closed-loop stability for a set of trajectories of interest. The presented technique is based on convex optimization and sparse identification methods, and thus avoids the problem of local minima and allows an efficient on-line implementation in real-world applications. The presence of noisy measurements and process disturbances is accounted for, too. The third contribution is a simulation study, concerned with the application of DFK to the challenging problem of control design for a class of airborne wind energy generators (see (Fagiano, Milanese & Piga 2010, Fagiano, Milanese & Piga available online)).

2 Problem formulation

Consider a nonlinear single-input discrete-time system S in state-space form:

$$x_{t+1} = g_o\left(x_t, u_t, e_t\right) \tag{1}$$

where $t \in \mathbb{Z}$ is the discrete time, $x_t \in X \subseteq \mathbb{R}^{n_x}$ is the state, $u_t \in U \subset \mathbb{R}$ is the input, $U \doteq [\underline{u}, \overline{u}]$ accounts for input saturation, and $e_t \in \mathbb{R}^{n_e}$ is a noise including both process and measurement disturbances.

Assumption 1 *The noise e_t is bounded:*

$$e_t \in B_\varepsilon \doteq \{e : \|e\|_\infty \leq \varepsilon\}, \quad \forall t \in \mathbb{Z}$$

for some $\varepsilon < \infty$. □

Assumption 2 *The function g_o is Lipschitz continuous with respect to u_t: $g_o\left(x_t, \cdot, e_t\right) \in \mathcal{F}\left(\gamma_g, U\right)$ for any $x_t \in X$ and any $e_t \in B_\varepsilon$, where*

$$\begin{aligned} &\mathcal{F}\left(\gamma, W\right) \doteq \{f : \|f\left(0\right)\|_\infty < \infty, \\ &\|f\left(z\right) - f\left(\widehat{z}\right)\|_\infty \leq \gamma \left\|z - \widehat{z}\right\|_\infty, \forall z, \widehat{z} \in W\} \end{aligned} \tag{2}$$

and $\|\cdot\|_\infty$ is the ℓ_∞ vector norm. □

Suppose that the function g_o defining the system (1) is unknown, but a set of noise-corrupted measurements is available:

$$\mathcal{D}^L \doteq \left\{\left(\widetilde{x}_{k+1}, \widetilde{x}_k\right), \widetilde{u}_k,\ k \in \mathcal{T}^L\right\} \tag{3}$$

where $\mathcal{T}^L \doteq \{-L+1, \ldots, 0\}$, $\|\widetilde{x}_k\|_\infty < \infty$ and $\widetilde{u}_k \in U$ for any $k \in \mathcal{T}^L$.

Let $\mathcal{X}^0 \subseteq X$ be a set of initial conditions of the system (1) and, for any given initial condition $x_0 \in \mathcal{X}^0$, let $\mathcal{R}\left(x_0\right) \subseteq \ell_\infty$ be a set of solutions of interest. The aim is to control the system (1) in such a way that, starting from any initial condition $x_0 \in \mathcal{X}^0$, the system solution $\boldsymbol{x} = (x_1, x_2, \ldots)$ tracks any reference signal $\boldsymbol{r} = (r_1, r_2, \ldots) \in \mathcal{R}\left(x_0\right)$. To accomplish this task, we use the feedback control scheme depicted in Figure 1, where S is the system (1), f is a controller, $r_t \in R$ is the reference, and $R \subseteq X$ is a bounded set where the trajectories of interest lie. Based on this closed-loop scheme, we investigate the following problem.

Problem 1 *Design a controller f such that the* tracking error

$$TE_t(f) \doteq \|r_t - x_t\|_\infty$$
$$= \|r_t - g_o(x_{t-1}, f(r_t, x_{t-1}), e_{t-1})\|_\infty$$

is "small" for all $t = 1, 2, \ldots$, and for any $(x_0, \boldsymbol{r}, \boldsymbol{e}) \in \mathcal{X}^0 \times \mathcal{R} \times \mathcal{B}_\varepsilon$, i.e. for any initial condition $x_0 \in \mathcal{X}^0$, any reference sequence $\boldsymbol{r} = (r_1, r_2, \ldots) \in \mathcal{R}(x_0)$, and any noise sequence $\boldsymbol{e} = (e_1, e_2, \ldots) \in \mathcal{B}_\varepsilon \doteq \{\varsigma = (\varsigma_1, \varsigma_2, \ldots) : \varsigma_t \in B_\varepsilon, \forall t\}$. □

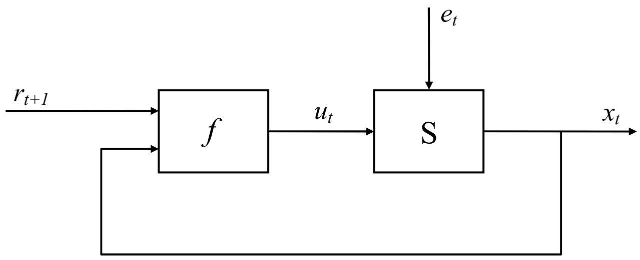

Fig. 1. Feedback control system.

In the following, we present a design approach to solve Problem 1, based on the inversion of g_o from data, along with a rigorous theoretical analysis on its stabilizing properties.

3 DFK control of nonlinear systems

Consider the function g_o defining the system (1). A controller f can be seen as a right-inverse function of g_o with respect to u_t, i.e. a function that, given the current state x_t and the value r_{t+1} desired for the state at time $t+1$, provides an input $u_t = f(r_{t+1}, x_t)$ such that $x_{t+1} \simeq r_{t+1}$.

For a right-inverse function f of g_o, we can define the following *point-wise inversion error*:

$$IE(f, r, x, e) \doteq \|r - g_o(x, f(r, x), e)\|_\infty$$

where $(r, x) \in \Omega \doteq R \times X \subset \mathbb{R}^{n_w}$, $n_w = 2n_x$, $e \in B_\varepsilon$, and $\|\cdot\|_\infty$ is the vector ℓ_∞ norm. We can also define the following *global inversion error*:

$$GIE(f) \doteq \|IE(f, \cdot, \cdot, \cdot)\|_\infty$$

where $\|\cdot\|_\infty$ is the functional L_∞ norm evaluated over $\Omega \times B_\varepsilon$.

Given a right-inverse function f with "small" global error $GIE(f)$, it holds that

$$r \simeq g_o(x, f(r, x), e) \tag{4}$$

for any $(r, x, e) \in \Omega \times B_\varepsilon$. For an invertible function g_o, f is also a left-inverse of g_o, since (4) implies that, if u is such that $r \simeq g_o(x, u, e)$, then $u \simeq f(r, x) = f(g_o(x, u, e), x)$. Thanks to these inversion properties, the function f could be used as the controller in the closed-loop system of Figure 1, providing a possible solution to the design Problem 1. In fact, if the closed-loop system is stable and $GIE(f)$ is "small", then a "small" tracking error $TE_t(f)$ can be obtained.

In order to formalize this idea, consider the following stability notion.

Definition 1 *A nonlinear system (possibly time-varying) with input u_t, state x_t, and noise $e_t \in B_\varepsilon$ is* finite-gain ℓ_∞ stable on $(\mathcal{X}^0, \mathcal{U})$ *if finite and non-negative λ and β exist such that*

$$\|\boldsymbol{x}\|_\infty \leq \lambda \|\boldsymbol{u}\|_\infty + \beta$$

for any $(x_0, \boldsymbol{u}, \boldsymbol{e}) \in \mathcal{X}^0 \times \mathcal{U} \times \mathcal{B}_\varepsilon$, where $\boldsymbol{x} = (x_1, x_2, \ldots)$, $\boldsymbol{e} = (e_1, e_2, \ldots) \in \mathcal{B}_\varepsilon$, $\boldsymbol{u} = (u_1, u_2, \ldots) \in \mathcal{U}$, and $\mathcal{U}$ is the input domain. □

Note that this finite-gain stability definition is more general than the standard one, which corresponds to the case $\mathcal{U} = \ell_\infty$, see e.g. (Khalil 1996). Moreover, if $\mathcal{U} = \{\boldsymbol{u} : \|\boldsymbol{u}\|_\infty < b_u\}$ for some $b_u < \infty$, then $\|\boldsymbol{x}\|_\infty \leq \lambda b_u + \beta$, which implies practical stability, see e.g. (Lakshmikantham, Leela & Martynyuk 1990). From this latter inequality we also have that, for any given time $T_h \in \mathbb{N}$, $\|x_t\|_\infty \leq \lambda b_u + \beta$, $t = 1, \ldots, T_h$, which implies finite-time stability for any time interval $[0, T_h]$, see e.g. (Mastellone, Dorato & Abdallah 2004, Amato, Ambrosino, Cosentino & Tommasi 2010).

Consider now the following stabilizability notion.

Definition 2 *Given a system of the form (1), define*

$$G(f, r_{t+1}, x_t, e_t) \doteq r_{t+1} - g_o(x_t, f(r_{t+1}, x_t), e_t).$$

The system is γ-stabilizable on $(\mathcal{X}^0, \mathcal{R})$ *if a $\gamma < \infty$ and a right-inverse function $f \in \mathcal{F}(\gamma, \Omega)$ exist such that*

$$G(f, r_{t+1}, \cdot, e_t) \in \mathcal{F}(\gamma_G, X), \quad \gamma_G < 1 \tag{5}$$

for all $t = 1, 2, \ldots$, and for any $(x_0, \boldsymbol{r}, \boldsymbol{e}) \in \mathcal{X}^0 \times \mathcal{R} \times \mathcal{B}_\varepsilon$. □

The γ-stabilizability condition (5) implies that a right-inverse function $f \in \mathcal{F}(\gamma, \Omega)$ exists such that the closed-loop system described by the difference equation $x_{t+1} = g_o(x_t, f(r_{t+1}, x_t), e_t)$ is finite-gain ℓ_∞ stable on $(\mathcal{X}^0, \mathcal{R})$. In fact, this equation can be written as $x_{t+1} = r_{t+1} - G(f, r_{t+1}, x_t, e_t)$, which corresponds to a closed-loop system with reference r_{t+1}, output x_{t+1} and negative feedback $G(f, r_{t+1}, x_t, e_t)$ (see the block diagram inside the dashed line in Figure 2 with $\hat{r}_{t+1} \to r_{t+1}$). As it can be easily seen applying the Small Gain Theorem (see e.g. (Khalil 1996)), closed-loop stability follows from the fact that $\gamma_G < 1$.

Stabilizability as defined here has analogies with flatness (Fliess, Lèvine, Martin & Rouchon 1995). However, the stabilizability condition is somewhat weaker than the flatness condition. To illustrate this fact, consider for example a linear system which is not completely controllable, but has a stable uncontrollable part. It is easy to verify that such a system is stabilizable according to Definition 2. On the other hand, this system is not flat, since flatness implies controllability.

The control design approach proposed in this paper requires the stabilizability of the system (1).

Assumption 3 *The system (1) is γ_o-stabilizable on $\left(\mathcal{X}^0, \mathcal{R}_o\right)$, where $\mathcal{X}^0 \subseteq X$ and $\mathcal{R}_o(x_0) \subseteq \ell_\infty$.* □

Under this assumption, we can define an optimal controller as follows.

Definition 3 *A right-inverse function $f_o : \Omega \to U$ is an* optimal controller *of the system (1) if*

$$f_o = \arg \min_{f \in \mathcal{S} \cap \mathcal{F}(\gamma_o, \Omega)} GIE\,(f) \tag{6}$$

where $\mathcal{S}$ is the set of all functions f satisfying the stabilizability condition (5). □

Thus, an optimal controller f_o is a right-inverse function which, among all functions stabilizing the closed-loop system, gives the minimum global inversion error.

Let us now consider a given optimal controller f_o. Clearly, this controller is not known since, as assumed in Section 2, the function g_o is not known (even for known g_o, the evaluation of f_o from (6) may be hard). In this paper, we propose an approach, called Direct FeedbacK (DFK) design, that uses an approximation $\widehat{f}$ of f_o derived from the available data (3), as the controller f in the closed-loop system of Figure 1. A suitable identification method for deriving the approximation $\widehat{f}$ will be proposed in Section 4. The remainder of this section is devoted to derive sufficient conditions under which an approximation $\widehat{f}$ stabilizes the closed-loop system, and to provide a bound on the resulting tracking error.

Define the following *residue function*:

$$\Delta\left(r_{t+1}, x_t\right) \doteq f_o\left(r_{t+1}, x_t\right) - \widehat{f}\left(r_{t+1}, x_t\right). \tag{7}$$

Choosing a Lipschitz continuous approximation $\widehat{f}$ and considering that f_o is Lipschitz continuous by definition, we have that Δ is also Lipschitz continuous. In particular,

$$\Delta \in \mathcal{F}\left(\gamma_\Delta, \Omega_\Delta\right) \tag{8}$$

for some $\gamma_\Delta < \infty$, where $\Omega_\Delta \subseteq \Omega$ is a compact convex.

Theorem 1 *Let Assumptions 1, 2 and 3 hold. Assume also that*

$$\gamma_g \gamma_\Delta < 1. \tag{9}$$

Then:

(i) The DFK closed-loop system

$$x_{t+1} = g_o\left(x_t, \widehat{f}\left(r_{t+1}, x_t\right), e_t\right) \tag{10}$$

is finite-gain ℓ_∞ stable on $\left(\mathcal{X}^0, \mathcal{R}^*\right)$, where $\mathcal{R}^* \doteq \{\boldsymbol{r} \in \mathcal{R}_o\left(\mathcal{X}^0\right) : \left(r_t, x_{t-1}\right) \in \Omega_\Delta,\ t = 1, 2, \ldots, \forall \boldsymbol{e} \in \mathcal{B}_\varepsilon\}$ and $\mathcal{R}_o\left(\mathcal{X}^0\right) \doteq \{\boldsymbol{r} \in \mathcal{R}_o\left(x_0\right),\ x_0 \in \mathcal{X}^0\}$.

(ii) The DFK closed-loop system tracking error is tightly bounded as

$$TE_t\left(\widehat{f}\right) \leq IE\left(f_o, r_t, x_{t-1}, e_{t-1}\right) + \gamma_g \left|\Delta\left(r_t, x_{t-1}\right)\right| \tag{11}$$

for all $t = 1, 2, \ldots,$ and for any $\left(x_0, \boldsymbol{r}, \boldsymbol{e}\right) \in \mathcal{X}^0 \times \mathcal{R}^* \times \mathcal{B}_\varepsilon$.

Proof. (i) Consider that

$$\begin{aligned} x_{t+1} &= g_o\left(x_t, \widehat{f}\left(r_{t+1}, x_t\right), e_t\right) \\ &= r_{t+1} - G(f_o, r_{t+1}, x_t, e_t) - F(f_o, \widehat{f}, r_{t+1}, x_t, e_t) \end{aligned} \tag{12}$$

where

$$\begin{aligned} G(f_o, r_{t+1}, x_t, e_t) &\doteq r_{t+1} - g_o\left(x_t, f_o\left(r_{t+1}, x_t\right), e_t\right) \\ F(f_o, \widehat{f}, r_{t+1}, x_t, e_t) &\doteq g_o\left(x_t, f_o\left(r_{t+1}, x_t\right), e_t\right) \\ &\quad - g_o\left(x_t, \widehat{f}\left(r_{t+1}, x_t\right), e_t\right). \end{aligned} \tag{13}$$

This shows that the system in Figure 1 is equivalent to the system in Figure 2 where q^{-1} is the time-shift operator: $q^{-1}x_{t+1} = x_t$.

By Definitions 2 and 3, the internal closed-loop system corresponding to the block diagram inside the dashed line in Figure 2 defined by

$$\begin{aligned} x_{t+1} &= \widehat{r}_{t+1} - G(f_o, r_{t+1}, x_t, e_t) \\ &= r_{t+1} - v_{t+1} - G(f_o, r_{t+1}, x_t, e_t) \end{aligned} \tag{14}$$

is finite-gain ℓ_∞ stable on $\left(\mathcal{X}^0, \ell_\infty\right)$. Indeed, it is a closed-loop system with reference $r_{t+1} - v_{t+1}$, output x_{t+1} and negative feedback $G(f_o, r_{t+1}, x_t, e_t)$. Since, by Assumption 3, $G(f_o, r_{t+1}, \cdot, e_t) \in \mathcal{F}\left(\gamma_G, X\right)$, $\gamma_G < 1$, for all $t = 1, 2, \ldots,$ any initial condition $x_0 \in \mathcal{X}^0$, any noise sequence $\boldsymbol{e} = (e_1, e_2, \ldots)$ such that $e_t \in B_\varepsilon$ $\forall t$, and any reference sequence $\boldsymbol{r} = (r_1, r_2, \ldots) \in \mathcal{R}_o\left(x_0\right)$, the Small Gain Theorem (see e.g. (Khalil 1996)) implies closed-loop finite-gain stability.

Moreover, from (14), we have that

$$\begin{aligned} \left\|x_{t+1}\right\|_\infty &\leq \left\|\widehat{r}_{t+1}\right\|_\infty + \left\|G(f_o, r_{t+1}, x_t, e_t)\right\|_\infty \\ &= \left\|\widehat{r}_{t+1}\right\|_\infty + \left\|r_{t+1} - g_o\left(x_t, f_o\left(r_{t+1}, x_t\right), e_t\right)\right\|_\infty \\ &\leq \left\|\widehat{r}_{t+1}\right\|_\infty + IE\left(f_o, r_{t+1}, x_t, e_t\right) \end{aligned}$$

where the last inequality follows from the definition of point-wise inversion error. It follows that, for any $\widehat{\boldsymbol{r}} =$

$(\widehat{r}_1, \widehat{r}_2, \ldots) \in \ell_\infty$,

$$\|\boldsymbol{x}\|_\infty \leq \|\widehat{\boldsymbol{r}}\|_\infty + GIE\left(f_o\right) \tag{15}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots)$.

On the other hand, since by Assumption 2 $g_o\left(x_t, \cdot, e_t\right) \in \mathcal{F}\left(\gamma_g, U\right)$ for any $x_t \in X$ and any $e_t \in B_\varepsilon$, we have that

$$\begin{aligned} \left\|F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)\right\|_\infty &= \| g_o\left(x_t, f_o\left(r_{t+1}, x_t\right), e_t\right) \\ &\quad - g_o\left(x_t, \widehat{f}\left(r_{t+1}, x_t\right), e_t\right) \Big\|_\infty \\ &\leq \gamma_g \left| f_o\left(r_{t+1}, x_t\right) - \widehat{f}\left(r_{t+1}, x_t\right) \right| = \gamma_g \left|\Delta\left(r_{t+1}, x_t\right)\right|. \end{aligned} \tag{16}$$

Moreover, $\Delta \in \mathcal{F}\left(\gamma_\Delta, \Omega_\Delta\right)$, and this implies that $\left|\Delta\left(r_{t+1}, x_t\right) - \Delta\left(r_{t+1}, 0\right)\right| \leq \gamma_\Delta \left\|x_t - 0\right\|_\infty$, i.e.

$$\left|\Delta\left(r_{t+1}, x_t\right)\right| \leq \gamma_\Delta \left\|x_t\right\|_\infty + \left|\Delta\left(r_{t+1}, 0\right)\right| \tag{17}$$

for any $(r_{t+1}, x_t) \in \Omega_\Delta$. Inequalities (16) and (17) yield

$$\left\|F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)\right\|_\infty \leq \gamma_g \gamma_\Delta \left\|x_t\right\|_\infty + \gamma_g \left|\Delta\left(r_{t+1}, 0\right)\right|$$

for any $(r_{t+1}, x_t) \in \Omega_\Delta$ and any $e_t \in B_\varepsilon$. Since R is a bounded set, and $\Delta \in \mathcal{F}\left(\gamma_\Delta, \Omega_\Delta\right)$, we have that $\gamma_g \left|\Delta\left(r_{t+1}, 0\right)\right|$ is bounded by some $\beta < \infty$:

$$\gamma_g \left|\Delta\left(r_{t+1}, 0\right)\right| \leq \beta,$$

for any $r_{t+1} \in R$. The following inequality thus hold for any $(r_{t+1}, x_t) \in \Omega_\Delta$, any $e_t \in B_\varepsilon$, and all $t = 1, 2, \ldots$:

$$\left\|F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)\right\|_\infty \leq \gamma_g \gamma_\Delta \left\|x_t\right\|_\infty + \beta.$$

This implies that, for any $\boldsymbol{x} = (x_1, x_2, \ldots)$ with $(r_{t+1}, x_t) \in \Omega_\Delta \; \forall t$,

$$\|\boldsymbol{v}\|_\infty \leq \gamma_g \gamma_\Delta \|\boldsymbol{x}\|_\infty + \beta \tag{18}$$

where $\boldsymbol{v} = (v_1, v_2, \ldots)$. Thus, the system inside the dot-dashed line in Figure 2 defined by

$$v_{t+1} = F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)$$

is finite-gain ℓ_∞ stable on $\{\boldsymbol{x} = (x_1, x_2, \ldots) : (r_{t+1}, x_t) \in \Omega_\Delta, \forall t\}$.

According to the Small Gain Theorem (see e.g. (Khalil 1996)), if (15) and (18) hold and $\gamma_g \gamma_\Delta < 1$, it follows that the overall closed-loop system in Figure 2 is finite-gain ℓ_∞ stable on $\mathcal{R}^*\left(x_0\right)$. This system is equivalent to the system (10), which is thus finite-gain ℓ_∞ stable on $\mathcal{R}^*\left(x_0\right)$ as well. This proves the first claim.

(ii) From (12), the tracking error is bounded as

$$\begin{aligned} TE_{t+1}\left(\widehat{f}\right) &\doteq \left\| r_{t+1} - g_o\left(x_t, \widehat{f}\left(r_{t+1}, x_t\right), e_t\right)\right\|_\infty \\ &\leq \left\|G(f_o, r_{t+1}, x_t, e_t) + F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)\right\|_\infty \\ &\leq \left\|G(f_o, r_{t+1}, x_t, e_t)\right\|_\infty + \left\|F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)\right\|_\infty \end{aligned} \tag{19}$$

for all $t = 1, 2, \ldots$ From (13) and from the point-wise inversion error definition, we have

$$\begin{aligned} &\left\|G(f_o, r_{t+1}, x_t, e_t)\right\|_\infty \\ &= \left\|r_{t+1} - g_o\left(x_t, f_o\left(r_{t+1}, x_t\right), e_t\right)\right\|_\infty \\ &= IE\left(f_o, r_{t+1}, x_t, e_t\right) \end{aligned} \tag{20}$$

for any $(r_{t+1}, x_t) \in \Omega$ and any $e_t \in B_\varepsilon$. From (16), $\left\|F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)\right\|_\infty$ is bounded as

$$\left\|F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)\right\|_\infty \leq \gamma_g \left|\Delta\left(r_{t+1}, x_t\right)\right| \tag{21}$$

for any $(r_{t+1}, x_t) \in \Omega$ and any $e_t \in B_\varepsilon$. The bound (11) follows from (19), (20), and(21).

Moreover, the bound (19) is tight since $F(f_o, \widehat{f}, r_{t+1}, x_t, e_t)$ is independent on $G(f_o, r_{t+1}, x_t, e_t)$. The point-wise inversion error $IE\left(f_o, r_{t+1}, x_t, e_t\right)$ in (20) is tightly bounded by $GIE\left(f_o\right)$, which is minimal due to Definition 3. The bound (21) is tight since, by definition of Lipschitz constant, γ_g is the smallest real number for which (21) holds for any $(r_{t+1}, x_t) \in \Omega$ and any $e_t \in B_\varepsilon$.

It follows that (11) provides a tight bound on $TE_t\left(\widehat{f}\right)$.

Note that tightness is here intended in a worst-case sense: these bounds are the tightest ones that can be derived on the basis of the available information. □

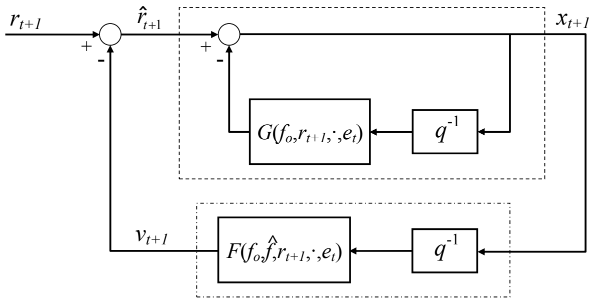

Fig. 2. Equivalent control system.

Theorem 1 can be interpreted as follows. If the system (1) is γ_o-stabilizable on $\left(\mathcal{X}^0, \mathcal{R}_o\right)$, then an optimal controller $f_o \in \mathcal{F}\left(\gamma_o, \Omega\right)$ exists such that the closed-loop system

$$x_{t+1} = g_o\left(x_t, f_o\left(r_{t+1}, x_t\right), e_t\right)$$

is finite-gain ℓ_∞ stable on $\left(\mathcal{X}^0, \mathcal{R}_o\right)$ and the tracking error is bounded as $TE_t\left(f_o\right) \leq IE\left(f_o, r_t, x_{t-1}, e_{t-1}\right)$, where $IE(f_o, r_t, x_{t-1}, e_{t-1})$ is the inversion error of f_o. However, f_o is not known and an approximation $\widehat{f}$ of it is used instead. If this approximation is sufficiently accurate on a set Ω_Δ, allowing us to have a Lipschitz constant γ_Δ satisfying the condition (9), then the closed-loop system

$$x_{t+1} = g_o\left(x_t, \widehat{f}(r_{t+1}, x_t), e_t\right) \tag{22}$$

is finite-gain ℓ_∞ stable on $(\mathcal{X}^0, \mathcal{R}^*)$, where $\mathcal{R}^*$ is the intersection of the set $\mathcal{R}_o$ with the set of all reference signals yielding a closed-loop solution such that $(r_t, x_{t-1}) \in \Omega_\Delta$ $\forall t$. Thus, the following requirements are needed to have finite-gain ℓ_∞ stability:

(1) The set Ω_Δ should be "well explored" by the data $(\widetilde{x}_{k+1}, \widetilde{x}_k)$, $k \in \mathcal{T}^L$ (a systematic method for assessing the quality of a given data set is provided in (Milanese & Novara 2007)). Using these data and a suitable identification method (see Section 4 below), a function $\widehat{f}$ providing an "accurate" approximation of f_o on this set can be found, so that γ_Δ is sufficiently small to satisfy the stability condition (9).

(2) The reference signals used for the controlled DFK system must belong to $\mathcal{R}_o$, i.e. they must be such that the function $G(f_o, r_{t+1}, x_t, e_t)$ has a Lipschitz constant $\gamma_G < 1$ $\forall t$. This requirement can be accomplished by using reference signals which are solutions (or approximate solutions) of the system (1), i.e. signals $\boldsymbol{r} = (r_1, r_2, \ldots)$ for which, at each time t, a $\widehat{u}_t \in U$ exists giving $r_{t+1} \simeq g_o(x_t, \widehat{u}_t, e_t)$.

If the closed-loop system (22) is finite-gain ℓ_∞ stable on $(\mathcal{X}^0, \mathcal{R}^*)$, then the tracking error is bounded according to (11), where $IE(f_o, r_t, x_{t-1}, e_{t-1})$ is the inversion error of f_o and $\gamma_g |\Delta(r_t, x_{t-1})|$ measures the deterioration due to using $\widehat{f}$ in place of f_o. Clearly, under the reasonable assumption that $IE(f_o, r_t, x_{t-1}, e_{t-1})$ is "small", we can conclude that the tracking error of the closed-loop system (10) is "small" if $\widehat{f}$ is an "accurate" approximation of f_o.

Remark 1 *Under the assumptions of Theorem 1, the controller f^* stabilizes the closed-loop system on the whole set $\mathcal{X}^0 \times \mathcal{R}^*$ and the bound (11) on the tracking error holds for any $x_0 \in \mathcal{X}^0$, $(r_t, x_{t-1}) \in \Omega_\Delta$ and any $e_t \in B_e$. The validity of the controller is thus maintained on the whole domain Ω_Δ and not only on the data used for design.*

Remark 2 *The Lipschitz constants γ_Δ and γ_g can be estimated by means of the validation method presented in (Milanese & Novara 2004) and summarized in Appendix. Thus, the stability condition (9) can actually be estimated. Note that, if γ_Δ and γ_g are under-estimated, the closed-loop system (10) may be unstable, even if supposed to be stable. No problems arise if γ_Δ and γ_g are over-estimated and condition (9) anyhow holds.*

Remark 3 *It can be shown that, if $\gamma_g \gamma_\Delta < (1 - \gamma_G)$ and $\Delta(0,0) = 0$, then the DFK closed-loop system is finite-gain ℓ_∞ stable on $(\mathcal{X}^0, \mathcal{R}^*)$* without bias. *That is, the closed-loop system is stable according to Definition 1 with $\beta = 0$. The inequality $\gamma_g \gamma_\Delta < (1 - \gamma_G)$ can also be interpreted as a* robust stability condition. *Indeed, this condition may ensure closed-loop stability (with bias) even if γ_Δ and γ_g are under-estimated.*

Remark 4 *The DFK approach has been proposed for the case where the system to be controlled is not known, but it can be applied also when this system is known, allowing a relatively simple control design. Indeed, if an accurate system model is available, the data (3) required for the design can be generated by simulation of the model equations.*

4 Design of almost-optimal sparse SM DFK controllers

In this section, following a Set Membership framework (Milanese & Vicino 1991), (Milanese, Norton, Lahanier & Walter 1996), (Kurzhanski & Veliov 1994), (Chen & Gu 2000), (Milanese & Novara 2004), (Milanese & Novara 2005*a*), (Milanese, Novara, Hsu & Poolla 2009), we present a technique to design DFK controllers with suitable optimality properties, from a set of data generated by the system (1). Then, we discuss some important aspects of the proposed technique, such as the basis function choice and the sparsity properties.

4.1 Design of almost-optimal sparse DFK controllers

Under the stability condition (9), the tracking error of a DFK control system is bounded as (11), where $IE(f_o, r_t, x_{t-1}, e_{t-1})$ is the inversion error of the optimal controller f_o and $\gamma_g |\Delta(r_t, x_{t-1})|$ measures the deterioration due to using an approximate controller $\widehat{f}$ in place of f_o. Since $IE(f_o, r_t, x_{t-1}, e_{t-1})$ depends on the intrinsic stabilizability properties of the system and cannot be modified, it is crucial to find a controller $\widehat{f}$ yielding a "small" deterioration $\gamma_g |\Delta(r_t, x_{t-1})|$. Clearly, this quantity is not known, since $\Delta \doteq f_o - \widehat{f}$, and f_o is not known. However, the following information is available.

Experimental information. The data (3) have been collected, which can be conveniently described as

$$\widetilde{u}_k = f_o(\widetilde{w}_k) + d_k, \quad k \in \mathcal{T}^L \tag{23}$$

where $\widetilde{w}_k \doteq (\widetilde{x}_{k+1}, \widetilde{x}_k)$ and $d_k \doteq \widetilde{u}_k - f_o(\widetilde{w}_k)$ is an unmeasured noise. □

Prior information on the noise d_k. Since the measurements $\widetilde{u}_k$ and $\widetilde{w}_k$ are bounded and $f_o \in \mathcal{F}(\gamma_o, \Omega)$, it follows that the noise d_k is bounded:

$$d_k \in B_\delta, \quad k \in \mathcal{T}^L \tag{24}$$

for some $\delta < \infty$ (B_δ is defined in Assumption 1). □

Prior information on the function Δ. From (8), $\Delta \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$ for some $\gamma_\Delta < \infty$ (dependent on $\widehat{f}$), where Ω_Δ is a compact convex set. $\square$

Given this information, we have that $f_o \in FIFS$, where $FIFS$ is the Feasible Inverse Function Set.

Definition 4 Feasible Inverse Function Set*:*

$$FIFS \doteq \{f = \text{sat}[f] = \widehat{f} + \Delta : \ \Delta \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta), \\ \widetilde{u}_k - f(\widetilde{x}_{k+1}, \widetilde{x}_k) \in B_\delta, \ k \in \mathcal{T}^L\}$$

where $\text{sat}[u] \doteq \max(\min(u, \overline{u}), \underline{u})$. $\square$

According to this definition, $FIFS$ is the set of all inverse functions consistent with the prior and experimental information. Hence, the tightest bound on the deterioration $\gamma_g |\Delta(r_t, x_{t-1})| = \gamma_g |f_o(r_t, x_{t-1}) - \widehat{f}(r_t, x_{t-1})|$ which can be derived on the basis of this information is given by $\gamma_g \sup_{f \in FIFS} |f_o(r_t, x_{t-1}) - \widehat{f}(r_t, x_{t-1})|$, leading to the following definition of *worst-case tracking error*:

$$WE_t\left(\widehat{f}\right) \doteq IE(f_o, r_t, x_{t-1}, e_{t-1}) + \gamma_g AE_t\left(\widehat{f}\right)$$

where

$$AE_t\left(\widehat{f}\right) \doteq \sup_{f \in FIFS} \left| f(r_t, x_{t-1}) - \widehat{f}(r_t, x_{t-1}) \right|$$

is called the *worst-case approximation error*. An optimal DFK controller is defined as an inverse function f_{op} which guarantees the closed-loop stability and minimizes the worst-case approximation error.

Definition 5 *An inverse function* f_{op} *is an* optimal DFK controller on $(\mathcal{X}^0, \mathcal{R}^*)$ *if:*
(i) $f_o - f_{op} \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$ *with* $\gamma_\Delta < 1/\gamma_g$.
(ii) $AE_t(f_{op}) = \inf_{\widehat{f}} AE_t\left(\widehat{f}\right)$*, for all* $t = 1, 2, \ldots,$ *and for any* $(x_0, \boldsymbol{r}, \boldsymbol{e}) \in \mathcal{X}^0 \times \mathcal{R} \times \mathcal{B}_\varepsilon$. $\square$

However, finding an optimal DFK controller may be hard or not convenient from a computational point of view, and sub-optimal solutions can be looked for. In particular, approximations that guarantee a degradation of at most 2 are often considered in the literature. These approximations are called almost-optimal (Traub, Wasilkowski & Woźniakowski 1988), (Milanese et al. 1996).

Definition 6 *An inverse function* f_{ao} *is an* almost-optimal DFK controller on $(\mathcal{X}^0, \mathcal{R}^*)$ *if:*
(i) $f_o - f_{ao} \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$ *with* $\gamma_\Delta < 1/\gamma_g$.
(ii) $AE_t(f_{ao}) \leq 2 \inf_{\widehat{f}} AE_t\left(\widehat{f}\right)$*, for all* $t = 1, 2, \ldots,$ *and for any* $(x_0, \boldsymbol{r}, \boldsymbol{e}) \in \mathcal{X}^0 \times \mathcal{R} \times \mathcal{B}_\varepsilon$. $\square$

An almost-optimal DFK controller can be obtained by means of the following algorithm, completely based on convex optimization.

Algorithm 1

(1) Take a set of Lipschitz continuous basis functions $\phi_i : \Omega \to U$, $i = 1, \ldots, N$, a $\delta > 0$, and a $0 < \eta \leq 2\delta$ (indications for properly choosing the basis functions, δ and η are given in Subsection 4.2 below).

(2) Estimate the Lipschitz constant γ_g of g_o by means of the validation method summarized in Appendix, using the data $(\widetilde{x}_k, \widetilde{u}_k)$, $k \in \mathcal{T}^L$. Choose a $\gamma_\Delta^s \simeq 1/\gamma_g$ such that $\gamma_\Delta^s < 1/\gamma_g$.

(3) Solve the optimization problem

$$\begin{aligned} & a^1 = \arg\min_{a \in \mathbb{R}^N} \|a\|_1 \\ & \text{subject to} \\ & (a) \ \|\widetilde{\boldsymbol{u}} - \Phi a\|_\infty \leq \delta \\ & (b) \ |\widetilde{u}_l - \widetilde{u}_k + [\phi(\widetilde{w}_k) - \phi(\widetilde{w}_l)] a| \\ & \quad \leq \gamma_\Delta^s \|\widetilde{w}_l - \widetilde{w}_k\|_\infty + \eta, \ l, k \in \mathcal{T}^L \end{aligned} \tag{25}$$

where $\widetilde{\boldsymbol{u}} \doteq (\widetilde{u}_1, \ldots, \widetilde{u}_L)$, $\phi(\widetilde{w}_k) \doteq [\phi_1(\widetilde{w}_k) \ \ldots \ \phi_N(\widetilde{w}_k)]$, and

$$\Phi \doteq \begin{bmatrix} \phi_1(\widetilde{w}_1) & \cdots & \phi_N(\widetilde{w}_1) \\ \vdots & \ddots & \vdots \\ \phi_1(\widetilde{w}_L) & \cdots & \phi_N(\widetilde{w}_L) \end{bmatrix}.$$

(4) Compute the coefficient vector a^* according to the following optimization problem:

$$\begin{aligned} & a^* = \arg\min_{a \in \mathbb{R}^N} \|\widetilde{\boldsymbol{u}} - \Phi a\|_\infty \\ & \text{subject to} \\ & (a) \ a_i = 0, \ i \in \overline{\text{supp}}(a^1) \\ & (b) \ |\widetilde{u}_l - \widetilde{u}_k + [\phi(\widetilde{w}_k) - \phi(\widetilde{w}_l)] a| \\ & \quad \leq \gamma_\Delta^s \|\widetilde{w}_l - \widetilde{w}_k\|_\infty + \eta, \ l, k \in \mathcal{T}^L \end{aligned} \tag{26}$$

where $\overline{\text{supp}}(a)$ is the complement of $\text{supp}(a)$, defined as $\{1, 2, \ldots, N\} \setminus \text{supp}(a)$, and $\text{supp}(a)$ is the support of a, i.e. the set of indices at which a is not null. $\square$

The DFK controller is defined by

$$f^*(r_{t+1}, x_t) = \text{sat}\left[\sum_{i=1}^N a_i^* \phi_i(r_{t+1}, x_t)\right]. \tag{27}$$

The rationale behind Algorithm 1 can be explained as follows. After the preliminary operations carried out in steps 1 and 2, the ℓ_1 norm of the coefficient vector a is minimized in step 3, leading to a sparse coefficient vector a^1, i.e. a vector with a "small" number of non-zero elements. Sparsity is important to allow an efficient on-line controller implementation in real-world applications (see Subsection 4.3 below for a more detailed discussion). Constraint (a) in (25) ensures the consistency between the measured data and the prior information on the noise affecting these data. Constraints (b) allow us to guarantee closed-loop stability when a sufficiently large number of data is used, see Theorem 3 below. At this point, a^1 does not give in general the minimum error $\|\widetilde{\boldsymbol{u}} - \Phi a\|_\infty$ among all the vectors with the same sparsity level. Then, a vector a^* is obtained in step 4 with the same sparsity as a^1, giving a minimum error $\|\widetilde{\boldsymbol{u}} - \Phi a\|_\infty$, and satisfying the constraints for closed-loop stability.

Note that Algorithm 1 is run in batch mode using the data set (3), and all the involved optimization problems are convex and can be easily solved in polynomial time. Thus, the algorithm does not require time-consuming computations and has not to be applied on-line. During operation, just the identified function f^* in (27) has to be evaluated.

Now, define the functions

$$\begin{aligned} \overline{\Delta}(w) &\doteq \min\left(\overline{\eta}(w), \min_{k\in\mathcal{T}^L} \overline{h}_k(w)\right) \\ \underline{\Delta}(w) &\doteq \max\left(\underline{\eta}(w), \max_{k\in\mathcal{T}^L} \underline{h}_k(w)\right) \end{aligned} \tag{28}$$

where $\overline{\eta}(w) = \overline{u} - f^*(w)$, $\quad \underline{\eta}(w) = \underline{u} - f^*(w)$,

$$\begin{aligned} \overline{h}_k(w) &\doteq \widetilde{u}_k - f^*(\widetilde{w}_k) + \delta + \gamma_\Delta^s \|w - \widetilde{w}_k\|_\infty \\ \underline{h}_k(w) &\doteq \widetilde{u}_k - f^*(\widetilde{w}_k) - \delta - \gamma_\Delta^s \|w - \widetilde{w}_k\|_\infty \end{aligned}$$

and $\overline{u}$ and $\underline{u}$ are the bounds on u_k defined below (1). The following theorem shows that, provided the closed-loop stability, f^* is an almost-optimal DFK controller. The theorem also provides a bound on the worst-case tracking error which can be effectively evaluated.

Theorem 2 *Let the optimization problem (25) be feasible. Let Assumptions 1, 2 and 3 hold. Let* $f_o - f^* \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$ *with* $\gamma_\Delta = \gamma_\Delta^s < 1/\gamma_g$. *Then:*
(i) The DFK controller f^* *is almost-optimal on* $(\mathcal{X}^0, \mathcal{R}^*)$.
(ii) The worst-case tracking error provided by f^* *is bounded as*

$$\begin{aligned} &WE_t(f^*) \le IE(f_o, r_t, x_{t-1}, e_{t-1}) + \xi_t \\ &\xi_t \doteq \gamma_g \max\left(\left|\overline{\Delta}(r_t, x_{t-1})\right|, \left|\underline{\Delta}(r_t, x_{t-1})\right|\right) \\ &\qquad \le 2\gamma_g \inf_{\widehat{f}} AE_t\left(\widehat{f}\right) \end{aligned} \tag{29}$$

for all $t = 1, 2, \ldots$, *and for any* $(x_0, \boldsymbol{r}, \boldsymbol{e}) \in \mathcal{X}^0 \times \mathcal{R}^* \times \mathcal{B}_\varepsilon$.

Proof. (i) If the optimization problem (25) is feasible, then the function f^* defined in (27) exists. Now consider that $f^* = f^* + \Delta$, with $\Delta = 0$. Obviously, if $\Delta = 0$, then $\Delta \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$, for any $\gamma_\Delta \ge 0$. Moreover, $f^* = \text{sat}[f^*]$ and, from (25) and (26), $\|\widetilde{\boldsymbol{u}} - f^*(\widetilde{w})\|_\infty \le \delta$, implying that $f^* \in FIFS$. This guarantees that

$$\begin{aligned} AE_t(f^*) &= \sup_{f\in FIFS} |f(r_t, x_{t-1}) - f^*(r_t, x_{t-1})| \\ &\le 2\inf_{\widehat{f}} \sup_{f\in FIFS} \left|f(r_t, x_{t-1}) - \widehat{f}(r_t, x_{t-1})\right| \end{aligned}$$

see (Traub et al. 1988), (Milanese et al. 1996). Then, the claim follows from the stability condition (9) and the definition of almost-optimal DFK controller.

(ii) It has been showed that $\Delta \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$. Then,

$$\Delta(w) \le \Delta(\widetilde{w}_k) + \gamma_\Delta \|w - \widetilde{w}_k\|_\infty, \ \forall w \in \Omega_\Delta$$

for any $k \in \mathcal{T}^L$, which yields

$$\Delta(w) \le \min_{k\in\mathcal{T}^L} \left(\Delta(\widetilde{w}_k) + \gamma_\Delta \|w - \widetilde{w}_k\|_\infty\right), \ \forall w \in \Omega_\Delta. \tag{30}$$

Now define $\varsigma_k \doteq \widetilde{u}_k - f^*(\widetilde{w}_k)$ and consider that, from (24),

$$\begin{aligned} &|\Delta(\widetilde{w}_k) - \varsigma_k| \\ &= |f_o(\widetilde{w}_k) - f^*(\widetilde{w}_k) - \widetilde{u}_k + f^*(\widetilde{w}_k)| \\ &= |f_o(\widetilde{w}_k) - \widetilde{u}_k| \le \delta, \quad k \in \mathcal{T}^L. \end{aligned}$$

This implies that $\Delta(\widetilde{w}_k) \le \varsigma_k + \delta$ for any $k \in \mathcal{T}^L$. Then, from (30), we have that

$$\begin{aligned} \Delta(w) &\le \min_{k\in\mathcal{T}^L} \left(\varsigma_k + \delta + \gamma_\Delta \|w - \widetilde{w}_k\|_\infty\right) \\ &\doteq \min_{k\in\mathcal{T}^L} \overline{h}_k(w), \ \forall w \in \Omega_\Delta. \end{aligned} \tag{31}$$

Moreover, since $\Delta(w) \doteq f_o(w) - f^*(w)$ and $f_o(w)$ is upperly saturated by $\overline{u}$, the following bound holds as well:

$$\Delta(w) \le \overline{u} - f^*(w) \doteq \overline{\eta}(w), \ \forall w \in \Omega_\Delta. \tag{32}$$

Inequalities (31) and (32) imply that the function $\overline{\Delta}$ is an upper bound of the unknown function Δ.

For given $w, \widehat{w} \in \Omega_\Delta$, let

$$\begin{aligned} i &\doteq \arg\min_{k\in\mathcal{T}^L} \left(\varsigma_k + \delta + \gamma_\Delta \|w - \widetilde{w}_k\|_\infty\right) \\ \widehat{i} &\doteq \arg\min_{k\in\mathcal{T}^L} \left(\varsigma_k + \delta + \gamma_\Delta \|\widehat{w} - \widetilde{w}_k\|_\infty\right). \end{aligned}$$

Then,

$$\begin{aligned} \overline{H}(w) &\doteq \min_{k\in\mathcal{T}^L} \overline{h}_k(w) \\ &= \varsigma_i + \delta + \gamma_\Delta \|w - \widetilde{w}_i\|_\infty \\ &\le \varsigma_{\widehat{i}} + \delta + \gamma_\Delta \left\|w - \widetilde{w}_{\widehat{i}}\right\|_\infty, \\ \overline{H}(\widehat{w}) &= \varsigma_{\widehat{i}} + \delta + \gamma_\Delta \left\|\widehat{w} - \widetilde{w}_{\widehat{i}}\right\|_\infty \end{aligned}$$

where the inequality follows from the fact that i is by definition the index minimizing $\varsigma_k + \delta + \gamma_\Delta \|w - \widetilde{w}_k\|_\infty$. These yield

$$\begin{aligned} &\overline{H}(w) - \overline{H}(\widehat{w}) \\ &\leq \gamma_\Delta \|w - \widetilde{w}_{\widehat{i}}\|_\infty - \gamma_\Delta \|\widehat{w} - \widetilde{w}_{\widehat{i}}\|_\infty \\ &\leq \gamma_\Delta \|w - \widehat{w}\|_\infty, \quad \forall w, \widehat{w} \in \Omega_\Delta, \end{aligned}$$

which shows that $\overline{H}$ is Lipschitz continuous with Lipschitz constant γ_Δ, i.e. that $\overline{H} \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$. Since $\overline{\Delta}(w) \doteq \min\left(\overline{\eta}(w), \overline{H}(w)\right)$ it follows that $\overline{\Delta} \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$ as well. Being $\overline{\Delta}$ an upper bound of Δ, this implies that

$$\sup_{\Delta \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)} \Delta(w) = \overline{\Delta}(w), \ \forall w \in \Omega_\Delta,$$

i.e. $\overline{\Delta}$ is the tightest upper bound of the unknown function Δ. Similarly, it can be shown that $\underline{\Delta}$ is the tightest lower bound of Δ. Under condition (9), the bound in (29) follows from (11) and the definition of ξ_t. □

Theorem 2 can be interpreted as follows. If the closed-loop system (22) is finite-gain ℓ_∞ stable on $\left(\mathcal{X}^0, \mathcal{R}^*\right)$ with $\widehat{f} = f^*$, then f^* is an almost-optimal DFK controller and the resulting tracking error is bounded as (11), where $IE(f_o, r_t, x_{t-1}, e_{t-1})$ is the inversion error of the optimal controller f_o and ξ_t is a tight bound on the deterioration due to using f^* in place of f_o. The fact that f^* is almost-optimal implies that ξ_t is very close to the minimum bound achievable. Moreover, ξ_t can be easily evaluated, showing that the bounds (11) and (29) are important, not only from a theoretical point of view, but also in practice, to quantify the tracking error deterioration due to using an approximation in place of the ideal controller f_o, see the example in Section 5.

These results hold under the stability condition $\gamma_\Delta < 1/\gamma_g$. Theorem 3 below shows that the inverse function f^* provided by Algorithm 1 satisfies this condition when the number of data L used to design f^* tends to infinity. The following assumption is needed to prove this theorem, requiring that the controller domain is "well explored" by the data $\widetilde{w}_k$, $k \in \mathcal{T}^L$.

Assumption 4 *The data set $\mathcal{D}_w^L \doteq \left\{\widetilde{w}_k,\ k \in \mathcal{T}^L\right\}$ is dense on Ω_Δ as $L \to \infty$. That is, for any $w \in \Omega_\Delta$ and any $\nu > 0$, an $L_\nu > 0$ and a $\widetilde{w}_k \in \mathcal{D}_w^{L_\nu}$ exist such that $\|w - \widetilde{w}_k\|_\infty \leq \nu$.* □

Another assumption is needed, regarding the noise affecting the data. Define $s_{lk} \doteq d_l - d_k$, and consider that

$$s_{lk} \in B_\eta, \quad l, k \in \mathcal{T}^L$$

where $\eta \leq 2\delta$ and δ is the bound on d_k, see (24). If d_k can take any value in the interval $[-\delta, \delta]$, then $\eta = 2\delta$. If d_k contains some systematic error (e.g. a constant term), then we may have $\eta < 2\delta$.

Assumption 5 *The noise $s_{lk} \doteq d_l - d_k$ is* tight. *That is, for any $\mu > 0$, an L_μ exists such that*

$$s_{l_1 k_1} \leq -\eta + \mu, \quad s_{l_2 k_2} \geq \eta - \mu$$

for some $l_1, k_1, l_2, k_2 \geq L_\mu$. □

This assumption essentially requires that the noise s_{lk} hits the bounds $-\eta$ and η with arbitrary closeness after a sufficiently long time. It must be remarked that this assumption is very weak, since no statistical information on s_{lk} is used. The definition of tight noise proposed here is a deterministic version of the probabilistic definition given in (Bai, Cho & Tempo 1998).

Theorem 3 *Let the optimization problem (25) be feasible for any $L > 0$. Let Assumptions 4 and 5 hold. Then, $f_o - f^* \in \mathcal{F}(\gamma_\Delta, \Omega_\Delta)$, where*

$$\lim_{L \to \infty} \gamma_\Delta = \gamma_\Delta^s < \frac{1}{\gamma_g}.$$

Proof. The data set $\mathcal{D}_w^L$ becomes dense in Ω_Δ as $L \to \infty$. Then, for any $w_1, w_2 \in \Omega_\Delta$ and any $\nu > 0$, an $L_\nu > 0$ and two points $\widetilde{w}_{i_1}, \widetilde{w}_{i_2} \in \mathcal{D}_w^{L_\nu}$ exist such that

$$\|w_1 - \widetilde{w}_{i_1}\|_\infty \leq \nu, \ \|w_2 - \widetilde{w}_{i_2}\|_\infty \leq \nu. \tag{33}$$

Since Δ is Lipschitz continuous with constant $\gamma_\Delta < \infty$, it follows that

$$\begin{aligned} &|\Delta(w_1) - \Delta(w_2)| = |\Delta(w_1) - \Delta(w_2) \\ &+\Delta(\widetilde{w}_{i_1}) - \Delta(\widetilde{w}_{i_1}) + \Delta(\widetilde{w}_{i_2}) - \Delta(\widetilde{w}_{i_2})| \\ &\leq |\Delta(\widetilde{w}_{i_1}) - \Delta(\widetilde{w}_{i_2})| + 2\gamma_\Delta \nu. \end{aligned} \tag{34}$$

From (23), we have that $f_o(\widetilde{w}_k) = \widetilde{u}_k - d_k$ for any k. Then,

$$\begin{aligned} &\Delta(\widetilde{w}_{i_1}) - \Delta(\widetilde{w}_{i_2}) = f_o(\widetilde{w}_{i_1}) - f_o(\widetilde{w}_{i_2}) \\ &+f^*(\widetilde{w}_{i_2}) - f^*(\widetilde{w}_{i_1}) \\ &= \widetilde{u}_{i_1} - \widetilde{u}_{i_2} - d_{i_1} + d_{i_2} + f^*(\widetilde{w}_{i_2}) - f^*(\widetilde{w}_{i_1}). \end{aligned} \tag{35}$$

Moreover, the constraints (b) in (25) and (26) imply that

$$\begin{aligned} &|\widetilde{u}_{i_1} - \widetilde{u}_{i_2} + f^*(\widetilde{w}_{i_2}) - f^*(\widetilde{w}_{i_1})| \\ &= |\widetilde{u}_{i_1} - \widetilde{u}_{i_2} + [\phi(\widetilde{w}_{i_2}) - \phi(\widetilde{w}_{i_1})]\, a| \\ &\leq \gamma_\Delta^s \|\widetilde{w}_{i_1} - \widetilde{w}_{i_2}\|_\infty + \eta. \end{aligned} \tag{36}$$

From (35) and (36), we have that

$$\begin{aligned} \Delta(\widetilde{w}_{i_1}) - \Delta(\widetilde{w}_{i_2}) &\leq \gamma_\Delta^s \|\widetilde{w}_{i_1} - \widetilde{w}_{i_2}\|_\infty + \eta + s_{i_2 i_1} \\ \Delta(\widetilde{w}_{i_1}) - \Delta(\widetilde{w}_{i_2}) &\geq -\gamma_\Delta^s \|\widetilde{w}_{i_1} - \widetilde{w}_{i_2}\|_\infty - \eta + s_{i_2 i_1} \end{aligned} \tag{37}$$

where $s_{i_2 i_1} \doteq d_{i_2} - d_{i_1}$.
Now, take a ν^1 such that

$$0 < \nu^1 < \min\left(\|w_1 - \widetilde{w}_{i_1}\|_\infty, \|w_2 - \widetilde{w}_{i_2}\|_\infty\right) < \nu.$$

Then, an $L_{\nu^1} > 0$ and two points $\widetilde{w}_{j_1}, \widetilde{w}_{j_2} \in \mathcal{D}_w^L$, with $L \geq L_{\nu^1}$ exist such that

$$\|w_1 - \widetilde{w}_{j_1}\|_\infty \leq \nu^1 < \nu, \; \|w_2 - \widetilde{w}_{j_2}\|_\infty \leq \nu^1 < \nu.$$

Repeating this argument an infinite number of times, we have that, for any $w_1, w_2 \in \Omega_\Delta$ and any $\nu > 0$, the number of couples $(\widetilde{w}_{i_1}, \widetilde{w}_{i_2})$, $(\widetilde{w}_{j_1}, \widetilde{w}_{j_2})$, ..., with $\widetilde{w}_{i_1}, \widetilde{w}_{i_2}, \widetilde{w}_{j_1}, \widetilde{w}_{j_2}, \ldots \in \mathcal{D}_w^L$, satisfying inequalities (37) tends to infinity as $L \to \infty$. This fact and Assumption 5 imply that, for any $\mu > 0$, a couple $(\widetilde{w}_{i_1}, \widetilde{w}_{i_2})$, with $\widetilde{w}_{i_1}, \widetilde{w}_{i_2} \in \mathcal{D}_w^L$, exists such that the first of inequalities (37) holds with $s_{i_2 i_1} \leq -\eta + \mu$ as $L \to \infty$. This implies that, for this couple,

$$\Delta(\widetilde{w}_{i_1}) - \Delta(\widetilde{w}_{i_2}) \leq \gamma_\Delta^s \|\widetilde{w}_{i_1} - \widetilde{w}_{i_2}\|_\infty + \mu.$$

Using inequalities (33), we obtain that

$$\Delta(\widetilde{w}_{i_1}) - \Delta(\widetilde{w}_{i_2}) \leq \gamma_\Delta^s \|w_1 - w_2\|_\infty + 2\gamma_\Delta^s \nu + \mu.$$

This inequality and (34) yield

$$\Delta(w_1) - \Delta(w_2) \leq \gamma_\Delta^s \|w_1 - w_2\|_\infty + 2\gamma_\Delta^s \nu + 2\gamma_\Delta \nu + \mu.$$

Analogously, it can be shown that

$$\Delta(w_1) - \Delta(w_2) \geq -\gamma_\Delta^s \|w_1 - w_2\|_\infty - 2\gamma_\Delta^s \nu - 2\gamma_\Delta \nu - \mu$$

and, consequently,

$$|\Delta(w_1) - \Delta(w_2)| \leq \gamma_\Delta^s \|w_1 - w_2\|_\infty + 2\gamma_\Delta^s \nu + 2\gamma_\Delta \nu + \mu. \quad (38)$$

Since this inequality holds for any $\nu, \mu > 0$, we have that

$$|\Delta(w_1) - \Delta(w_2)| \leq \gamma_\Delta^s \|w_1 - w_2\|_\infty$$

for any $w_1, w_2 \in \Omega_\Delta$. The claim follows from the fact that $\gamma_\Delta^s < 1/\gamma_g$. □

Remark 5 *If γ_g is excessively over-estimated in step 2 of Algorithm 1, γ_Δ^s may be chosen too small and, consequently, the optimization problems (25) and (26) may result to be not feasible. On the other hand, if γ_g is excessively under-estimated, the closed-loop system (10) may result to be unstable, even if supposed to be stable. In order to improve the DFK control robustness and to prevent stability problems in situations where γ_g is under-estimated, the constant γ_Δ^s in step 2 of Algorithm 1 can be chosen such that $\gamma_\Delta^s < (1 - \gamma_G)/\gamma_g$, (see Remark 3), where γ_G can be used as a design parameter, allowing us to tune the DFK control rubustness level.*

4.2 Choice of basis functions and noise bound

One important step required by any parametric identification method is the choice of the basis functions ϕ_i. This choice can be carried out considering the numerous options available in the literature (e.g. Gaussian, sigmoidal, wavelet, polynomial, trigonometric), see (Sjöberg, Zhang, Ljung, Benveniste, B.Delyon, Glorennec, Hjalmarsson & Juditsky 1995). Gaussian basis functions are widely used, defined as

$$\phi_i(w) = e^{-\left\|Q(w - \widetilde{w}_i)\right\|^2} \quad (39)$$

where $Q \in \mathbb{R}^{n_w \times n_w}$ is a diagonal matrix where the kth element is proportional to the width of the function along the dimension k. These functions are universal approximators, see e.g. (White 1991), and will be used in the example presented in Section 5.
An inappropriately chosen family of functions can force the retention of many basis functions by the identification algorithm, and this may lead to an increased inverse model complexity and, in some situation, to a large tracking error or even to closed-loop instability. The available prior information can be used in order to address this important issue. Suppose that the bound δ is (approximately) known from some prior information available on the noises acting on the system. If the optimization problem (25) in Algorithm 1 is feasible and a^* is sparse, it may be concluded that the basis functions have been correctly chosen, since a small number of them is able to "explain" the measured data. If the optimization problem (25) is feasible and a^* is not sparse, it may be guessed that the basis function choice is not appropriate, since using a number of basis functions about equal to (or greater than) the number of data may lead to overfitting problems. If the optimization problem (25) is not feasible, it may be certainly concluded that the basis function choice is not appropriate. In the case that no prior information on the noises is available, the value of δ should be taken slightly larger than the minimum value for which the optimization problem (25) is feasible and then, if necessary, tuned by means of a trial and error procedure. Once δ has been chosen, the parameter η required in steps 3 and 4 of Algorithm 1 can be easily chosen considering that this parameter is a bound on the noise $s_{lk} \doteq d_l - d_k$. This noise is certainly bounded by 2δ, but it can also be smaller than 2δ, e.g. in the presence of systematic errors. Thus, a reasonable choice is $\eta = 2\delta - \rho$, for some small $\rho > 0$.

4.3 Sparsity properties

In Section 4, an ℓ_1 algorithm for sparse approximation from data has been presented, which provides almost-optimal controllers for nonlinear systems. The reason why a sparse approximation has been looked for is twofold. First, a sparse function is easy to implement on real-time processors, which may have limited memory and computational capacity. Second, sparse approximations are able to provide good accuracy on new data by limiting well-known issues such as over-fitting and the curse of dimensionality. In the remainder of this subsection, we briefly discuss the notion of sparse function and explain the connection with the proposed ℓ_1 algorithm.

A *sparse* function is a linear combination of many basis functions, where the vector of linear combination coefficients is sparse, i.e. it has only a few non-zero elements. The *sparsity* of a vector a is typically measured by the ℓ_0 quasi-norm, defined as the number of its non-zero elements. Sparse identification can thus be performed by looking for a coefficient vector of the basis function linear combination with a "small" ℓ_0 quasi-norm. However, the ℓ_0 quasi-norm is a non-convex function and its minimization is in general an NP-hard problem. Two main approaches are commonly adopted to deal with this issue: convex relaxation and greedy algorithms (Tropp 2004), (Fuchs 2005), (Tropp 2006), (Donoho, Elad & Temlyakov 2006). In convex relaxation, a suitable convex function, e.g. the ℓ_1 norm, is minimized instead of the ℓ_0 quasi-norm (Fuchs 2005), (Tropp 2006), (Donoho et al. 2006). In greedy algorithms, the sparse solution is obtained iteratively, (Tropp 2004). Algorithm 1 presented in Section 4 is essentially an improved ℓ_1 algorithm: in step 1, an optimization problem is solved, where the ℓ_0 quasi-norm is replaced by the ℓ_1 norm, and additional constraints for closed-loop stability are used. In step 2, a vector a^* is obtained with the same support of a^1, giving minimum $\|\widetilde{\boldsymbol{u}} - \Phi a\|_\infty$, and satisfying the constraints for closed-loop stability.

5 DFK control of a power kite

5.1 Kite-energy control problem

The Kite-energy technology aims to harvest high altitude wind energy by using tethered flexible wings (power kites), connected to the ground by means of two lines, made of strong composite fiber and wound around two drums, kept at the ground level and linked to reversible electric motors (Fagiano, Milanese & Piga 2010). The system composed by the kite, the lines, the on-board sensors, the drums, the generators and the control hardware is named Kite Steering Unit (KSU). A kite energy generator prototype has been constructed at Politecnico di Torino (in collaboration with local companies), which has also been used for naval propulsion (Fagiano, Milanese, Razza & Gerlero 2010, *http://www.kitenav.com/* n.d.), (Fagiano, Milanese, Razza & Bonansone 2012). The KSU can be employed in different configurations to generate energy (see e.g. (Fagiano 2009), (Fagiano et al. available online) for details). In the so-called KE-yoyo configuration, the KSU is fixed with respect to the ground and energy is generated by continuously repeating a two-phase cycle, in which the lines are unrolled by strong pulling forces, thus generating power, and then rolled back using low pulling forces. In the KE-carousel configuration, the line length is kept constant and energy is produced by exploiting the motion of the KSU towed by the kite along a fixed path on the ground. In whatever configuration is used, the kite has to be controlled to fly on figure-eight paths in crosswind conditions (see Figure 4), which maximize the pulling forces on the lines and hence the generated electrical power. However, such paths are unstable and cannot be tracked without a proper feedback control, see e.g. (Ilzhöfer, Houska & Diehl 2007).

One of the key components of the Kite-energy system is therefore the control system, whose task is guiding the kite in order to generate the maximum amount of power, while at the same time satisfying operational constraints, since the wing has to be kept above a minimal height from the ground and line wrapping has to be avoided. The design of the kite control system has been carried out in (Ilzhöfer et al. 2007), (Fagiano 2009) and (Fagiano, Milanese & Piga 2010) by applying Nonlinear Model Predictive Control (NMPC) techniques with quite good results. However, these techniques rely on an accurate model of the system, which in this case is hard to derive due to the various uncertainties involved in the dynamics of a real power kite. The DFK approach, not requiring the knowledge of any model, has been adopted here instead.

5.2 Application of DFK control to Kite-energy

The model described in (Fagiano 2009) and (Fagiano, Milanese & Piga 2010) has been used as the "real" system, with the parameter values indicated in Table 1.

Table 1
Model parameters

m	5	Kite mass (kg)
A	10	Characteristic area (m^2)
d_l	0.0035	Diameter of a single line (m)
ρ_l	970	Line density (kg/m^3)
$C_{D,l}$	1	Line drag coefficient
α_0	3.5	Base angle of attack (°)
ρ	1.2	Air density (kg/m^3)
r	50	Line length (m)
d	5	Distance between the lines' hang points on the kite (m)
Δ_t	0.1	Sample time (s)

In this model, a Cartesian coordinate system (X, Y, Z) is considered (see. Fig. 3), where the X axis is aligned with the nominal wind speed vector direction. The wind speed vector is represented as $\vec{W}_l = \vec{W}_0 + \vec{W}_T$, where $\vec{W}_0$ is the nominal wind speed, supposed to be known and expressed in (X, Y, Z) coordinates as

$$\vec{W}_0 = \Big(W_X(Z),\ 0,\ 0 \Big) \tag{40}$$

and $W_X(Z)$ is a known function which describes the variation of wind speed with respect to the altitude Z (see e.g. (Archer & Jacobson 2005)). The term $\vec{W}_T$ may have components in all directions and is not supposed to be known, accounting for wind unmeasured turbulence. The kite position can be expressed as a function of its distance r from the origin and of the two angles θ and φ, as depicted in Fig. 3, which also shows the three unit vectors e_θ, e_φ and e_r of a local coordinate system centered at the kite center of gravity. The variable r is also the length of the lines, supposed to be straight.

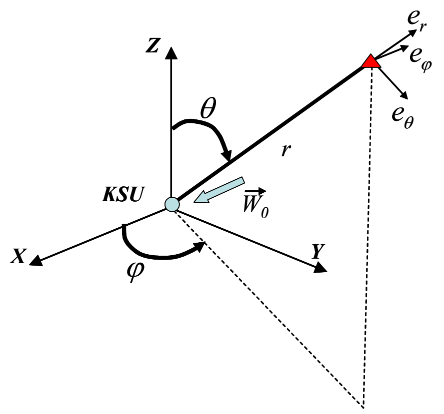

Fig. 3. Model scheme of a KE–yoyo.

For simplicity, a fixed value of r has been used in this example, but the DFK approach proposed here can be used without significant modifications also with a variable line length r. The considered system model is described by a set of differential equations (see (Fagiano 2009), (Fagiano, Milanese & Piga 2010)) of the form

$$\dot{z}(\tau) = g_o^c\left(x(\tau), u(\tau), \vec{W}_T(\tau)\right) \tag{41}$$

where $\tau \in \mathbb{R}$ is the continuous time, $x(\tau) = (\theta(\tau), \dot{\theta}(\tau), \varphi(\tau), \dot{\varphi}(\tau))$ is the state of the system, $\vec{W}_T$ is the wind turbulence speed, $u(\tau) = \arcsin(\Delta l/d)$ is the command input, Δl is the difference between the lengths of the two lines, d is the distance between the attachment points of the two lines on the kite.

DFK design has been performed supposing that the model (41) is unknown, but a set of noise-corrupted data can be acquired through preliminary experiments. In order to generate these data, a real-time simulator of the model (41) has been developed, where the input $u(\tau)$ can be chosen by means of a joystick. Using this simulator and considering a sampling period $T_s = 0.5$ s, a set of $L = 2000$ measurements $\widetilde{u}_k = u(T_s k)$, $\widetilde{x}_k = x(T_s k)$, $k = -L+1, \ldots, 0$, has been generated. In this simulation, the kite has been "manually" piloted by means of the joystick in such a way to cover the region $\varphi \in [-1, 1]$ rad, $\theta \in [0.5, 1.5]$ rad, without falling down. The following wind shear model (see (40)) has been used:

$$W_X(Z) = 6.7 \log\left(\frac{Z}{0.1}\right) / \log\left(\frac{50}{0.1}\right) \tag{42}$$

where the nominal wind speed W_X is about 6 m/s at 30 m of altitude. The wind turbulence has been simulated by adding to the nominal wind $\vec{W}_0$ a random Gaussian vector of zero mean and standard deviation std$(w(\tau))$ $= (1.5, 1.5, 1.5)$ m/s. The measurements of each component of the state $\widetilde{x}_k$ have been corrupted by a white Gaussian noise having a noise to signal standard deviation ratio of 3%. The main reason for considering Gaussian noises is to show that our design method, which relies on the assumption of bounded noise, can provide satisfactory performances even in cases where the noises are not deterministic but stochastic. Moreover, for 10 of the collected data points, the value of $\widetilde{u}_k$ has been significantly changed in order to simulate the presence of outliers.

A discrete-time almost-optimal controller of (41) has been designed by means of Algorithm 1 (Subsection 4.1) from the generated data, where the outliers have been removed through the algorithm described in the Appendix. This controller is given by

$$u_t = f^*(r_{t+1}, x_t)$$

where $t = 1, 2, \ldots$ is the discrete time, $u_t = u(T_s t)$, $x_t = x(T_s t)$, and f^* is a sparse function of the form (27). $N = L = 2000$ Gaussian basis functions of the form (39) have been used, where the diagonal elements of Q have been chosen by means of Lemma 2 in (Milanese & Novara 2004). The value $\delta = 2.65$ has been chosen in step 1 of Algorithm 1 (slightly larger than the minimum value for which the optimization problem (25) was feasible). Consequently, the value $\eta = 5.2 \simeq 2\delta$ has been taken. In step 2, the Lipschitz constant γ_g has been estimated using the validation procedure summarized in appendix, and the value $\gamma_g = 1.5$ has been obtained. Then, the value $\gamma_\Delta^s = 0.6 < 1/\gamma_g$ has been chosen. All the optimization problems in Algorithm 1 have been solved using the CVX package (Grant & Boyd 2010).

The control system depicted in Figure 1 has been implemented in Simulink, where $e_t = \vec{W}_T(T_s t)$, $\widehat{f} = f^*$, and S is the kite system (41). The control system has then been tested using a reference signal corresponding to a periodic orbit having, in the (φ, θ)-plane, a figure-eight shape, see Figure 4. This signal has been generated by manually piloting the kite as close as possible to a figure-eight path and then smoothing the resulting measured signals. The figure-eight path has been chosen considering the geometric and physical properties of the kite and the limitations imposed by the available space. As previously discussed, this kind of orbit is optimal in terms of traction force maximization, but is unstable without feedback control.

For comparison, a NMPC controller has also been implemented. This controller, operating with a sampling time τ_s (not necessarily equal to T_s), is based on applying at each time step $l\tau_s$, $l = 1, 2, \ldots$, the following algorithm:

(1) Compute a k-step prediction $\hat{X}_l(\bar{u}) = (\hat{x}_{l+1}, \ldots, \hat{x}_{l+k})$

by iterating a discretization of the "true" system equations (41) over the time interval $[l\tau_s, l\tau_s + k\tau_s]$ assuming a constant input $u(\tau) = \bar{u}$ and a wind speed $\vec{W}_T = \Big(W_X(Z),\, 0,\, 0 \Big)$.

(2) Find the command input by solving the optimization problem $u_l^* = \arg\min_{\bar{u}\in[-20,20]} \left\| R_l - \hat{X}_l(\bar{u}) \right\|$, where $R_l = (r_{l+1}, \ldots, r_{l+k})$, r_l is the reference signal obtained from the above figure-eight orbit and $\|\cdot\|$ is the matrix 2-norm. This optimization problem, though non-convex, can be solved quite efficiently by means of a gridding technique ($\bar{u}$ is a scalar variable). □

After extensive simulations, the sampling time $\tau_s = 0.1$ s and the prediction step horizon $k = 1$ have been selected for the NMPC controller, since providing the best results in terms of tracking error. It has also been tried to perform the k-step prediction in step (1) by integration of the "true" system equations (41) over the time interval $[l\tau_s, l\tau_s + k\tau_s]$. However, the same tracking accuracy has been obtained, at the expense of a significantly higher computational time. A grid of 20 points uniformly distributed in the interval [-20,20] has been used in step (2): no significant improvements have been observed considering a larger number of points.

The Root Mean Square tracking errors provided by the DFK and NMPC control systems have been evaluated for two levels of turbulence: weak turbulence (std($w(\tau)$) $= (0.5, 0.5, 0.5)$ m/s) and strong turbulence (std($w(\tau)$) $= (2, 2, 2)$ m/s). These errors have been computed as $RMS_i = RMS\,(r_{i,t} - x_{i,t})$, where i indicates the ith vector component and

$$RMS\,(z_t) \doteq \sqrt{\frac{1}{200}\sum_{t=1}^{200} z_t^2}.$$

A Monte Carlo simulation consisting of 1000 trials has been carried out, where all the above operations have been repeated for each trial.

In Tables 2 and 3, the RMS error averaged over the 1000 trials are reported for the two control systems (the average values are denoted with $\overline{RMS}$). In Figure 4, the orbit of the kite driven by the DFK controller obtained in one trial is compared in the (φ, θ)-plane to the reference for the two wind strength levels. From these results, it can be concluded that the DFK controllers are able to yield an accurate tracking, even for strong turbulence, achieving performances similar to those obtained by the NMPC controller, which has been designed using the exact knowledge of the system dynamics (a situation that rarely occurs in practice). It can also be noted that, among the 2000 basis functions, only 63 have been selected by Algorithm 1 in average, yielding controllers f^* whose evaluation is very "fast" and can be easily performed on-line. Note that the average times required for computing the DFK and NMPC controllers at a given point resulted to be about 1 ms and 9 ms, respectively, on a standard PC equipped with two Xeon 2GHz processors.

The bound ξ_t measuring the deterioration due to using an approximation in place of the ideal (unknown) controller f_o has also been evaluated for the 1000 realizations, yielding an average Root Mean Square value $\overline{RMS}\,(\xi_t) = 0.029$ in both the cases of weak and strong turbulence. This value is quite small compared to the average Root Mean Square of the system state, which resulted to be $\overline{RMS}(\|x_t - \overline{x}\|_\infty) = 0.543$, $\overline{x} = \frac{1}{200}\sum_{t=1}^{200} x_t$, showing that the approximate controllers f^* do not give a significant performance degradation with respect to the ideal controllers f_o.

It must be remarked that the situation simulated in this example is quite realistic: in a first phase, the kite is "manually" piloted in order to generate data; in a second phase, the data are used for model identification and/or control design; in a third phase, the kite is automatically piloted by the designed controller.

Table 2
Root mean square tracking errors, wind speed std=0.5 m/s.

Controller	$\overline{RMS}_1$	$\overline{RMS}_2$	$\overline{RMS}_3$	$\overline{RMS}_4$
DFK	0.009	0.010	0.011	0.009
NMPC	0.033	0.025	0.010	0.007

Table 3
Root mean square tracking errors, wind speed std=2 m/s.

Controller	$\overline{RMS}_1$	$\overline{RMS}_2$	$\overline{RMS}_3$	$\overline{RMS}_4$
DFK	0.028	0.024	0.046	0.032
NMPC	0.057	0.040	0.027	0.021

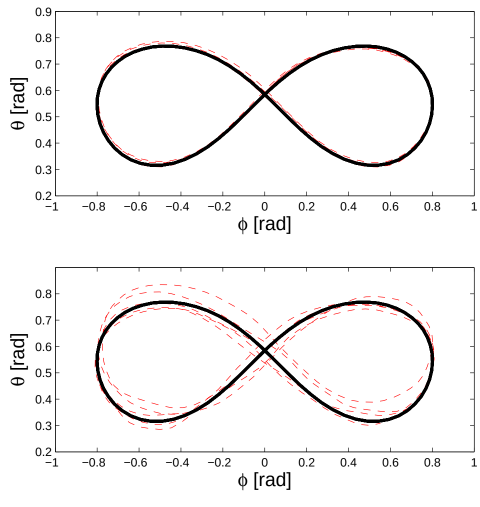

Fig. 4. Kite orbit. Above: weak wind. Below: strong wind. Bold (black) line: reference. Dashed (red) line: kite trajectory.

6 Conclusions

In this paper, an approach called Direct FeedbacK (DFK) design has been proposed, for the direct design from data of control laws for nonlinear state-space systems. The approach overcomes relevant problems typical of the standard design methods, such as modeling errors, non-trivial parameter identification, non-convex optimization and difficulty in nonlinear control design, and allows an efficient on-line controller implementation in real-world applications. A simulation study has also been presented, regarding DFK control of a power kite used for high altitude wind energy conversion. This study shows that the proposed technique may be used with satisfactory results in quite challenging applications, involving complex unstable nonlinear systems.

Acknowledgment: This research has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n. PIOF-GA-2009-252284 - Marie Curie project "Innovative Control, Identification and Estimation Methodologies for Sustainable Energy Technologies".

References

Abreu, G. C. M. D., Teixeira, R. L. & Ribeiro, J. F. (2000), A neural network-based direct inverse control for active control of vibrations of mechanical systems, *in* 'Proceedings of the sixth Brazilian Symposium on Neural Networks', pp. 107–112.

Amato, F., Ambrosino, R., Cosentino, C. & Tommasi, G. D. (2010), 'Input-output finite time stabilization of linear systems', *Automatica* **46**(9), 1558–1562.

Anuradha, D. B., Reddy, G. P. & Murthy, J. S. N. (2009), Direct inverse neural network control of a continuous stirred tank reactor (cstr), *in* 'Proceedings of the International MultiConference of Engineers and Computer Scientists', Hong Kong.

Archer, C. L. & Jacobson, M. Z. (2005), 'Evaluation of global wind power', *J. Geophys. Res.* **110, D12110**.

Bai, E., Cho, H. & Tempo, R. (1998), 'Convergence properties of the membership set', *Automatica* **34**(10), 1245–1249.

Cabrera, J. & Narendra, K. S. (1999), 'Issues in the application of neural networks for tracking based on inverse control', *IEEE Transaction on Automatic Control* **44**(11), 2007–2027.

Campi, M. & Savaresi, S. (2006), 'Direct nonlinear control design: The virtual reference feedback tuning (VRFT) approach', *IEEE Transactions on Automatic Control* **51**(1), 14–27.

Chen, J. & Gu, G. (2000), *Control-Oriented System Identification: An H_∞ Approach*, John Wiley & Sons, New York.

Donoho, D., Elad, M. & Temlyakov, V. (2006), 'Stable recovery of sparse overcomplete representations in the presence of noise', *Information Theory, IEEE Transactions on* **52**(1), 6 – 18.

Fagiano, L. (2009), Control of Tethered Airfoils for High–Altitude Wind Energy Generation, PhD thesis, Politecnico di Torino, Italy. Available on–line: http://lorenzofagiano.altervista.org/docs/PhD_thesis_Fagiano_Final.pdf.

Fagiano, L., Milanese, M. & Piga, D. (2010), 'High-altitude wind power generation', *IEEE Transactions on Energy Conversion* **25**(1), 168–180.

Fagiano, L., Milanese, M. & Piga, D. (available online), 'Optimization of airborne wind energy generators', *International Journal of Robust and Nonlinear Control* .

Fagiano, L., Milanese, M., Razza, V. & Bonansone, M. (2012), 'High-altitude wind energy for sustainable marine transportation', *IEEE Transactions on Intelligent Transportation Systems* **13**(2), 781–791.

Fagiano, L., Milanese, M., Razza, V. & Gerlero, I. (2010), Power kites for naval propulsion, *in* 'American Control Conference 2010', Baltimore, MD.

Fliess, M., Lèvine, J., Martin, P. & Rouchon, P. (1995), 'Flatness and defect of non-linear systems: introductory theory and examples', *Internat. J. Control* **61**, 1327–1361.

Fuchs, J. (2005), 'Recovery of exact sparse representations in the presence of bounded noise', *Information Theory, IEEE Transactions on* **51**(10), 3601 –3608.

Grant, M. & Boyd, S. (2010), 'CVX: Matlab software for disciplined convex programming, version 1.21', http://cvxr.com/cvx.

Hsu, K., Claassen, M., Novara, C., Khargonekar, P., Milanese, M. & Poolla, K. (2005), Non-parametric identification of static nonlinearities in a general interconnected system, *in* 'IFAC World Congress', Prague, Czech Republic.

Hsu, K., Vincent, T., Novara, C., Milanese, M. & Poolla, K. (2005), Identification of nonlinear maps in interconnected systems, *in* '44th IEEE Conference on Decision and Control and European Control Conference', Seville, Spain.

http://www.kitenav.com/ (n.d.).

Ilzhöfer, A., Houska, B. & Diehl, M. (2007), 'Nonlinear MPC of kites under varying wind conditions for a new class of large-scale wind power generators', *International Journal of Robust and Nonlinear Control* **17**, 1590–1599.

Khalil, H. (1996), *Nonlinear Systems*, Prentice Hall.

Kurzhanski, A. & Veliov, V. (1994), *Modeling techniques for uncertain systems*, Birkhäuser.

Lakshmikantham, V., Leela, S. & Martynyuk, A. (1990), *Practical Stability of Nonlinear Systems*, World Scientific, Singapore.

Levin, A. & Narendra, K. (1996), 'Control of nonlinear dynamical systems using neural networks. ii. observability, identification, and control', *IEEE Transaction on Neural Networks* **7**(1), 30–42.

Mastellone, S., Dorato, P. & Abdallah, C. (2004), Finite-time stability of discrete-time nonlinear systems: Analysis and design, *in* '43rd IEEE Conference on Decision and Control', Atlantis, Paradise Island, Bahamas.

Milanese, M., Norton, J., Lahanier, H. P. & Walter, E. (1996), *Bounding Approaches to System Identification*, Plenum Press.

Milanese, M. & Novara, C. (2004), 'Set membership identification of nonlinear systems', *Automatica* **40/6**, 957–975.

Milanese, M. & Novara, C. (2005*a*), 'Model quality in identification of nonlinear systems', *IEEE Transactions on Automatic Control* **50**(10), 1606–1611.

Milanese, M. & Novara, C. (2005*b*), 'Set membership prediction of nonlinear time series', *IEEE Transactions on Automatic Control* **50**(11), 1655–1669.

Milanese, M. & Novara, C. (2007), 'Computation of local radius of information in SM-IBC identification of nonlinear systems', *Journal of Complexity* **23**, 937 951.

Milanese, M., Novara, C., Hsu, K. & Poolla, K. (2009), 'The filter design from data (FD2) problem: Nonlinear Set Membership approach', *Automatica* **45**(10).

Milanese, M. & Vicino, A. (1991), 'Optimal algorithms estimation theory for dynamic systems with set membership uncertainty: an overview', *Automatica* **27**, 997–1009.

Norgaard, M., Ravn, O., Poulsen, N. & Hansen, L. K. (2000), *Neural Networks for Modeling and Control of Dynamic Systems*, Springer.

Sjöberg, J., Zhang, Q., Ljung, L., Benveniste, A., B.Delyon, Glorennec, P., Hjalmarsson, H. & Juditsky, A. (1995), 'Nonlinear black-box modeling in system identification: a unified overview', *Automatica* **31**, 1691–1723.

Traub, J. F., Wasilkowski, G. W. & Woźniakowski, H. (1988), *Information-Based Complexity*, Academic Press, Inc.

Tropp, J. (2004), 'Greed is good: algorithmic results for sparse approximation', *Information Theory, IEEE Transactions on* **50**(10), 2231 – 2242.

Tropp, J. (2006), 'Just relax: convex programming methods for identifying sparse signals in noise', *Information Theory, IEEE Transactions on* **52**(3), 1030 –1051.

White, H. (1991), 'Universal approximation using radial-basis-function networks', *Neural Computation* **3**, 246–257.

7 Appendix: Nonlinear SM validation procedure

In this appendix, the validation theory of (Milanese & Novara 2004) is summarized. This theory is important within the DFK approach for estimating the Lipschitz constants γ_Δ and γ_g appearing in Theorems 1 and 2. The formulation presented here considers a generic function $\mathfrak{f}$ with Lipschitz constant γ, but can be applied with minor modifications to estimate the constants γ_g of g_o and γ_Δ of Δ.

Suppose that a set of data $(\widetilde{w}_k, \widetilde{z}_k)$, $k \in \mathcal{T}^L$ is available, described by

$$\widetilde{z}_k = \mathfrak{f}(\widetilde{w}_k) + e_k, \quad k \in \mathcal{T}^L$$

where e_k is a noise (not necessarily the same as the one in (1)), $\mathfrak{f} : W \to Z$, $W \subseteq \mathbb{R}^{n_w}$ and $Z \subseteq \mathbb{R}$. Assume that $e_t \in B_\varepsilon$ for some $\varepsilon > 0$ and that $\mathfrak{f} \in \mathcal{F}(\gamma, W)$. Under this assumption, we have that $\mathfrak{f} \in FFS$, where FFS is the Feasible Function Set.

Definition 7 Feasible Function Set*:*

$$FFS \doteq \{f \in \mathcal{F}(\gamma, W) : \widetilde{z}_k - f(\widetilde{w}_k) \in B_\varepsilon,\ k \in \mathcal{T}^L\}.$$

□

According to this definition, FFS is the set of all functions consistent with prior assumptions and data. As typical in any identification/estimation theory, the problem of checking the validity of prior assumptions arises. The only thing that can be actually done is to check if prior assumptions are invalidated by the data, evaluating if no function exists consistent with data and assumptions, i.e. if FFS is empty.

Definition 8 *Prior assumptions are validated if $FFS \neq \emptyset$.* □

The following result provides necessary and sufficient conditions for prior assumption validation. Define the function $\overline{f}(\gamma, w) \doteq \min_{k \in \mathcal{T}^L} (\widetilde{z}_k + \varepsilon + \gamma \|w - \widetilde{w}_k\|)$.

Theorem 4
i) A necessary condition for prior assumptions to be validated is: $\overline{f}(\gamma, \widetilde{w}_k) \geq \widetilde{z}_k - \varepsilon,\ k \in \mathcal{T}^L$.
ii) A sufficient condition for prior assumptions to be validated is: $\overline{f}(\gamma, \widetilde{w}_k) > \widetilde{z}_k - \varepsilon,\ k \in \mathcal{T}^L$.

Proof. Minor modifications of the proof of Theorem 1 in (Milanese & Novara 2004). □

The validation Theorem 4 can be used for assessing the value of the Lipschitz constant γ so that the sufficient condition holds. Suppose that ε has been chosen by means of any criterion (e.g. using some prior knowledge on the noises in the case that $\mathfrak{f} = g_o$ or following the indications given in Subsection 4.2 when $\mathfrak{f} = \Delta$; the dispersion function defined in (Hsu, Vincent, Novara, Milanese & Poolla 2005), (Hsu, Claassen, Novara, Khargonekar, Milanese & Poolla 2005) can be used in general to estimate ε). The constant

$$\gamma^{\min} \doteq \inf_{\overline{f}(\widehat{\gamma}, \widetilde{w}_k) > \widetilde{z}_k - \varepsilon,\ k \in \mathcal{T}^L} \widehat{\gamma} \tag{43}$$

represents the minimum Lipschitz constant for which the prior assumptions are validated. A reasonable estimate of γ is thus a value slightly larger than $\gamma^{\min}$. Note that the evaluation of $\gamma^{\min}$ is quite simple, as shown by the examples in (Milanese & Novara 2004) and (Milanese & Novara 2005*b*).

Theorem 4 can also be used to eliminate eventual outliers from the data set $\mathcal{D}^L \doteq \{(\widetilde{w}_k, \widetilde{z}_k), k \in \mathcal{T}^L\}$. Indeed, suppose that the sufficient condition of the theorem is satisfied for a subset of data $\mathcal{D}^{L_m} \subset \mathcal{D}^L$ with $L_m \approx L$ but it is not satisfied for another subset $\mathcal{D}^{L_f} \subset \mathcal{D}^L$ with $L_f \ll L$. Suppose also that $\overline{f}(\gamma, \widetilde{w}_k) \ll \widetilde{z}_k - \varepsilon$ for $(\widetilde{w}_k, \widetilde{z}_k) \in \mathcal{D}^{L_f}$. Then, it can be concluded that the data in $\mathcal{D}^{L_f}$ are outliers, and consequently these outliers can be eliminated from $\mathcal{D}^L$. More specifically, this operation can be carried out by means of the following algorithm:

(1) For $k, l \in \mathcal{T}^L, l > k$, compute

$$\gamma_{kl} = \begin{cases} (\widetilde{z}_k - \widetilde{z}_l - 2\varepsilon)/\|\widetilde{w}_k - \widetilde{w}_l\|, & \widetilde{z}_k > \widetilde{z}_l + 2\varepsilon \\ (\widetilde{z}_l - \widetilde{z}_k - 2\varepsilon)/\|\widetilde{w}_k - \widetilde{w}_l\|, & \widetilde{z}_l > \widetilde{z}_k + 2\varepsilon \\ 0, & \text{otherwise.} \end{cases}$$

(2) For several values of $\hat{\gamma}$ ranging in the interval $[\min_{k,l}(\gamma_{kl}), \max_{k,l}(\gamma_{kl})]$, evaluate the number N_u of γ_{kl} values for which $\gamma_{kl} \geq \hat{\gamma}$. For increasing $\hat{\gamma}$, the number N_u drastically decreases until $\hat{\gamma}$ reaches a certain critical value γ^*, then it decreases with a significantly slower slope.
(3) Choose as the estimate of γ a value γ^{est} slightly larger than γ^*.
(4) Eliminate from the data set all points $(\widetilde{w}_k, \widetilde{z}_k)$ such that $\overline{f}(\gamma^{est}, \widetilde{w}_k) < \widetilde{z}_k - \varepsilon$. □