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Adaptive Model Predictive Control for Constrained Time Varying Systems

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***Abstract*— An approach to design feedback controllers for discrete-time, uncertain, linear time-varying systems subject to constraints is proposed. Building on previous contributions in the framework of time-invariant systems, in each sampling period a two-step procedure is carried out. In the first step, a set of linear models that are consistent with past input-output data and prior assumptions is built and refined. This set is guaranteed to contain also the true system dynamics if the considered working assumptions are valid. The time-varying nature of the plant is captured by assuming known bounds on the rate of change of the model parameters in time. In the second step, a robust finite-horizon optimal control problem is formulated and solved. The resulting optimal control sequence guarantees that the outputs of all possible plants inside the model set satisfy the operational constraints. The approach is showcased in numerical simulations on a three-tank system.**

I. Introduction

The capability to explicitly account for constraints on inputs and outputs and to incorporate information on future disturbance and reference signals are among the main reasons for the success of Model Predictive Control (MPC) in industrial applications [13], [14]. Duall, adaptive and learning-based MPC, i.e. approaches where the model derivation/identification step is considered together with the control computation step and possibly carried out on-line are receiving increased research attention recently. To date, there are several contributions that differ in terms of system dynamics (linear or nonlinear), uncertainty characterization (stochastic or unknown-but-bounded), and model identification scheme (off-line or on-line/adaptive), see e.g. [1], [2], [4], [5], [6], [8], [9], [10], [11], [12], [16], [17], [18], [19]. The main motivations for these works are the difficulty to derive and identify models based on physical principles for complex processes, the increasing widespread availability of measured data, and the want to derive MPC approaches that can automatically tune and adapt in presence of uncertain or time-varying plant dynamics.

In the described context, set-membership (SM) approaches (see e.g. [3]) are being adopted by several researchers for the model identification phase, since they can provide, in addition to a nominal model of the plant, a quantification of the associated uncertainty, which can be exploited for robust control design. Examples of contributions exploiting SM techniques are [5], [10], [11] and [16]. In [16], we proposed the use of SM identification to derive an adaptive MPC approach for uncertain linear time-invariant systems, with guaranteed constraint satisfaction and tracking performance. Here, we extend the approach to the case of time-varying systems. As in [16], a two-step procedure is carried out in each sampling period. In the first step, a set of linear models that are consistent with past input-output data and prior assumptions is built and refined. This set is guaranteed to contain also the true system dynamics if the considered working assumptions are valid. The time-varying nature of the plant is captured by assuming known bounds on the rate of change of the model parameters in time. In the second step, a robust finite-horizon optimal control problem is formulated and solved, where we also predict all possible future changes of the model set. The procedure is implemented on-line in a receding horizon fashion. We illustrate the approach in numerical simulations on a three-tank system. For a detailed theoretical analysis of the proposed approach, the interested reader is referred to [15].

II. Problem Formulation

We consider a discrete, linear time varying (LTV), multiple input, multiple output (MIMO) system with n_u inputs and n_y outputs. The system is known to be stable, but the exact dynamics and the way they change over time are not known. We denote the vector of control inputs at time step $t \in \mathbb{Z}$ by $u(t) = [u_1(t), \ldots, u_{n_u}(t)]^T$, where $u_i(t) \in \mathbb{R}$, $i = 1, \ldots, n_u$ are the individual plant inputs at the time step t and ^{T} stands for the matrix transpose operator. In addition, we denote the vector of plant outputs by $y(t) = [y_1(t), \ldots, y_{n_y}(t)]^T$, where $y_j(t) \in \mathbb{R}$, $j = 1, \ldots, n_y$ are the individual plant outputs. At each time step, the dynamic relation between the inputs and the outputs can be described by a finite impulse response model of the following form:

$$y_j(t) = H_j^T(t)\varphi(t) + d_j(t),\; j = 1, \ldots, n_y, \tag{1}$$

where $\varphi(t) \in \mathbb{R}^{n_u m}$ is the regressor vector formed by the m past control inputs:

$$\varphi(t+1) = [u(t-1)^T u(t-2)^T \ldots u(t-m)^T], \tag{2}$$

and each of the vectors $H_j, j = 1, \ldots, n_y$ contains the impulse response coefficient that describe the influence of all of the inputs to the output j.

We further denote the vector of output disturbances by $d(t) = [d_1(t), \ldots, d_{n_y}(t)]^T$, where $d_j(t) \in \mathbb{R}$, $j = 1, \ldots, n_y$ in (1) accounts for the contribution of the exogenous disturbances and of unmodeled dynamics to the output j at time step t. By defining the matrix $H(t) \in \mathbb{R}^{n_y \times n_u m}$ as $H(t) \doteq \left[H_1(t), \ldots, H_{n_y}(t)\right]^T$, the dependence of the plant

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output on the control inputs and the disturbance vectors at time step t can be written as:

$$y(t) = H(t)\varphi(t) + d(t). \tag{3}$$

Not that although a finite impulse response model is used, systems with infinite impulse response can be treated in this framework by embedding the contribution of the unmodelled dynamics due to truncation of the impulse response into disturbance term $d(t)$.

The measured output available for feedback control is corrupted by noise. In particular, the vector of measured plant outputs is denoted by:

$$\tilde{y}(t) = y(t) + v(t),$$

where $v(t) = [v_1(t), \dots, v_{n_y}(t)]^T$ and $v_j(t)$, $j = 1, \dots, n_y$ are the individual measurement noise terms that affect each of the measured plant outputs.

We consider the following assumption about the disturbance and noise signals.

Assumption 1: (Prior assumption on disturbance and noise) d and v are bounded as:

$$\begin{array}{lcl} |d_j(t)| & \leq & \epsilon_{d_j} \\ |v_j(t)| & \leq & \epsilon_{v_j} \end{array}, \forall t \in \mathbb{Z}, \forall j = 1, \dots, n_y, \tag{4}$$

where ϵ_{d_j} and ϵ_{v_j} are positive scalars.

We also consider two additional assumptions on the system to be controlled.

Assumption 2: (Assumption on the bounds on parameter rate of change)

$$H(t) - H(t-1) = \Delta H(t) \in \mathcal{D}, \forall t \in \mathbb{Z}, \tag{5}$$

where

$$\mathcal{D} \doteq \left\{ \Delta H \in \mathbb{R}^{n_y \times n_u m} : K_j \Delta H_j \leq l_j, j = 1, \dots, n_y \right\}, \tag{6}$$

and $K_j \in \mathbb{R}^{n_{\Delta_j} \times n_u m}$ and $l_j \in \mathbb{R}^{n_{\Delta_j}}, j = 1, \dots, n_y$ are known matrices and vectors that each define n_{Δ_j} linear inequalities forming nonempty, closed and convex sets, i.e. polytopes.

Assumption 3: (Assumption on the bounds on parameter values)

The plant model parameters belong to the following parameter set at all times: $H(t) \in \Omega, \forall t \in \mathbb{Z}$, with

$$\Omega \doteq \left\{ H \in \mathbb{R}^{n_y \times n_u m} : A_{j0} H_j \leq b_{j0}, j = 1, \dots, n_y \right\}, \tag{7}$$

where the inequalities in (7) should be interpreted as element-wise and each matrix $A_{j0} \in \mathbb{R}^{r_{j0} \times n_u m}$ and vector $b_{j0} \in \mathbb{R}^{r_{j0}}$ define a polytope with r_{j0} faces.

The control objective is to track a given output reference and reject disturbances over a (possibly very long) time horizon T ($T \gg m$), while enforcing input and output constraints. Assuming that the control inputs $u(l)$, $l = -m + 1, \dots, -1$ are known, we formalize the described control objective with the following optimization problem:

$$\min_{u(0),\dots,u(T)} \sum_{t=0}^{T} \left(y(t) - y_{\text{des}}(t) \right)^T Q \left(y(t) - y_{\text{des}}(t) \right) + u(t)^T S u(t) + \Delta u(t)^T R \Delta u(t) \tag{8}$$

Subject to, $\forall t \in [0, T]$

$$\begin{array}{lcl} Cu(t) & \leq & g \\ L\Delta u(t) & \leq & f \\ Ey(t) & \leq & o \end{array} \tag{9}$$

where $y_{\text{des}}(t) \in \mathbb{R}^{n_y}$ is the desired output reference, $Q \in \mathbb{R}^{n_y \times n_y}$, $S \in \mathbb{R}^{n_u \times n_u}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive semi-definite weighting matrices selected by the control designer, and $\Delta u(t) = u(t) - u(t-1)$ is the rate of change of the control input. The element-wise inequalities in (9) define convex sets through the matrices $C \in \mathbb{R}^{n_i \times n_u}$, $L \in \mathbb{R}^{n_{\Delta_u} \times n_u}$, $E \in \mathbb{R}^{n_o \times n_y}$ and the vectors $g \in \mathbb{R}^{n_i}$, $f \in \mathbb{R}^{n_{\Delta_u}}$, $o \in \mathbb{R}^{n_o}$, where n_i, n_{Δ_u} and n_o are the number of linear constraints on the inputs, input rates and outputs, respectively. We assume that the set defining the constraints on $\Delta u(t)$ contains the origin and that the constraint set of $u(t)$ is compact, which are assumptions that are satisfied in most practical problems.

III. Adaptive control algorithm

In order to approximately minimize the cost function (8) without the exact knowledge of the system dynamics, while at the same time satisfying the input and output constraints (9), we propose the use of a receding horizon control policy that relies on set membership identification to keep track of a set of all possible model parameters (feasible parameter set) that are consistent with the initial assumptions and the collected measurements. This model set is employed to predict the plant behavior with the related uncertainty intervals.

In the next subsections, we first describe the recursive set membership identification algorithm and then the finite horizon optimal control problem to be solved at each time step.

A. Recursive set membership identification algorithm

Each new measurement collected at time step t, defines a set to which the parameter matrix $H(t)$ is guaranteed to belong:

$$\mathcal{S}_t(t) \doteq \left\{ \begin{array}{ll} H \in \mathbb{R}^{n_y \times n_u m} : & \left| H_j^T \varphi(t) - \tilde{y}_j(t) \right| \leq \epsilon_{d_j} + \epsilon_{v_j}, \\ & j = 1, \dots, n_y \end{array} \right\} \tag{10}$$

The set $\mathcal{S}_t(t)$ is formed by n_y slabs that are defined by the regressor vector $\varphi(t)$ and the output measurements $\tilde{y}_j(t), j = 1, \dots, n_y$ collected at time step t. More generally, we define the set $\mathcal{S}_k(t)$ as the set defined by the regressor and output measurement vectors at time step $k \leq t$, i.e. $\varphi(k)$ and $\tilde{y}(k)$, that is guaranteed to contain the model parameter matrix $H(t)$ at time step t as:

$$\mathcal{S}_k(t) \doteq \left\{ \begin{array}{l} H \in \mathbb{R}^{n_y \times n_u m} : \\ -\epsilon_{d_j} - \epsilon_{v_j} + (t-k)\underline{\vartheta}_j(k) \leq H_j^T \varphi(k) - \tilde{y}_j(k), \\ H_j^T \varphi(k) - \tilde{y}_j(k) \leq \epsilon_{d_j} + \epsilon_{v_j} + (t-k)\overline{\vartheta}_j(k), \\ j = 1, \dots n_y \end{array} \right\}, \tag{11}$$

where each of the bounds $\underline{\vartheta}_j(k) \in \mathbb{R}$ and $\overline{\vartheta}_j(k) \in \mathbb{R}$, $j = 1, \ldots, n_y$ is given as the solution of the following two linear programs (LPs):

$$\begin{array}{rl} \underline{\vartheta}_j(k) \doteq & \min\limits_{x \in \mathbb{R}^{n_u m}} \varphi^T(k)x \\ \overline{\vartheta}_j(k) \doteq & \max\limits_{x \in \mathbb{R}^{n_u m}} \varphi^T(k)x \\ & \text{Subject to:} \\ & K_j x \leq l_j. \end{array} \tag{12}$$

Based on the definition of the set $\mathcal{S}_k(t)$ in (11) and the Assumptions 1, 2 and 3, we define the feasible parameter set at time step t, denoted by $\mathcal{F}(t)$, as the set that is guaranteed to contain all model parameter matrices at time step t, i.e. $H(t)$, that are consistent with the initial assumptions and the output measurements collected up to time step t:

$$\mathcal{F}(t) \doteq \Omega \cap \left(\bigcap_{k=1,\ldots,t} \mathcal{S}_k(t) \right). \tag{13}$$

According to Assumption 3, the set Ω is defined through polytopic constraints on the rows of the parameter matrix $H(t)$. Moreover, the sets $\mathcal{S}_k(t)$, $k = 1, \ldots, t$ are also defined through linear inequality constraints on the rows of the matrix $H(t)$. Therefore, the feasible parameter set $\mathcal{F}(t)$, defined as the intersection of all these sets, is also given by polytopic constraints on the rows of the model parameter matrix $H(t)$ and it can be represented as follows:

$$\mathcal{F}(t) = \left\{ H \in \mathbb{R}^{n_y \times n_u m} : A_j(t) H_j \leq b_j(t) \right\}, \tag{14}$$

where each of the matrices and vectors $A_j(t) \in \mathbb{R}^{r_j(t) \times n_u m}, b_j(t) \in \mathbb{R}^{r_j(t)}$, $j = 1, \ldots, n_y$ define $r_j(t)$ linear inequalities.

In order to use the defined feasible parameter set $\mathcal{F}(t)$ for the real-time computation of control inputs, a recursive update strategy is needed. To this end, we note that the matrix $A_j(t)$ can be created from the matrix $A_j(t-1), j = 1, \ldots, n_y$ by appending two rows formed by the regressor vector at time step t, $\varphi(t)$ and that the vector $b_j(t)$ can be formed from the vector $b_j(t-1), j = 1, \ldots, n_y$, by first adding the terms that account for the possible changes of the plant model with respect to the previous time step and then by appending two new rows that define the constraints related to the newly collected output measurement $\tilde{y}_j(t), j = 1, \ldots, n_y$:

$$A_j(t) = \begin{bmatrix} A_j(t-1) \\ -\varphi^T(t) \\ \varphi^T(t) \end{bmatrix}, \; b_j(t) = \begin{bmatrix} b_j(t-1) + \Delta b_j(t-1) \\ -\tilde{y}_j(t) + \epsilon_{d_j} + \epsilon_{v_j} \\ \tilde{y}_j(t) + \epsilon_{d_j} + \epsilon_{v_j} \end{bmatrix}, \tag{15}$$

where the vectors $\Delta b_j(t-1) \in \mathbb{R}^{r_j(t-1)}, j = 1, \ldots, n_y$ are computed as:

$$\Delta b_j(t-1) = \left[\mathbf{0}_{r_{jo}}, -\underline{\vartheta}_j(0), \overline{\vartheta}_j(0), \ldots, -\underline{\vartheta}_j(t-1), \overline{\vartheta}_j(t-1) \right]^T \tag{16}$$

with $\mathbf{0}_{r_{jo}} \in \mathbb{R}^{r_{j0}}$ denoting a vector of r_{jo} zeros.

The use of the recursive equation (15) to update the matrices $A_j(t)$ and vectors $b_j(t)$, $j = 1, \ldots, n_y$, would generally result in a continuous increase of their dimension $r_j(t)$, $j = 1, \ldots, n_y$ with each new output measurement, which would lead to a computationally intractable problem over time. For this reason, we only keep track of the constraints that are generated by the last M output measurements, where $M \in \mathbb{R}^+$ is a control design parameter, plus the r_{j0} inequalities given by the bounds (7) (see Assumption 3). The proposed recursive update of the feasible parameter set is described in Algorithm 1.

Algorithm 1 Recursive algorithm to update the feasible parameter set

1) At time step $t = 0$, for $j = 1, \ldots, n_y$, set $A_j(0) = A_{j0}$, $b_j(0) = b_{j0}$;
2) At time step $t > 0$, calculate the regressor vector $\varphi(t)$ according to (2) and take the measurement vector $\tilde{y}(t)$;
3) For $j = 1, \ldots, n_y$, calculate $\underline{\vartheta}_j(t)$ and $\overline{\vartheta}_j(t)$ by solving linear programs as in (12);
4) For $j = 1, \ldots, n_y$ form the matrix $A_j(t)$ and the vector $b_j(t)$ from $A_j(t-1)$ and $b_j(t-1)$ according to (15);
5) For $j = 1, \ldots, n_y$, if $r_j(t) > r_{j0} + M$, remove the $r_{j0} + 1$ and if needed $r_{j0} + 2$ row from the matrix $A_j(t)$ and vector $b_j(t)$, such that after removal it holds that $r_j(t) \leq r_{j0} + M$;
6) Set $t = t + 1$, go to 2).

In addition to the feasible parameter set, the proposed identification algorithm also provides a nominal model of the plant at each time step. The latter is given by a matrix $H_c(t) \in \mathbb{R}^{n_y \times n_u m}$, $H_c = [H_{c,1}, \ldots H_{c,n_y}]^T$, where the vectors $H_{c,j}(t) \in \mathbb{R}^{n_u m}$, $j = 1, \ldots, n_y$ can be calculated by solving s convex program that aims to find the point inside the feasible parameter set $\mathcal{F}(t)$ that is closest to the nominal model at the previous time step (i.e. $H_c(t-1)$):

$$\begin{array}{l} \min\limits_{H_{c,j}(t), j=1,\ldots,n_y} \sum\limits_{j=1}^{n_y} \| H_{c,j}(t-1) - H_{c,j}(t) \| \\ \text{Subject to:} \\ A_j(t) H_{cj}(t) \leq b_j(t), \; \forall j = 1, \ldots, n_y. \end{array} \tag{17}$$

Initially, at time step $t = 0$, the matrix $H_c(0)$ can be taken as an arbitrary nonzero point inside the set Ω. In (17), typical employed norms are $1-$, $\infty-$ and $2-$norms.

B. Finite horizon optimal control problem

Let $u(k|t)$, $k \in [t, t+N-1]$, $N \geq m$, be the candidate future control moves, where the notation $k|t$ indicates the prediction at step $k \geq t$ given the information at the current step t. For brevity, we collect these decision variables in vector $U \doteq [u(t|t)^T \ldots u(t+N-1|t)^T]^T$. We also define the vectors of future input increments $\Delta u(k|t)$, $k \in [t, t+N-1]$ as:

$$\Delta u(k|t) = \begin{cases} u(t|t) - u(t-1) & \text{if } k = t \\ u(k|t) - u(k-1|t) & \text{otherwise.} \end{cases}$$

Moreover, we define the future regressor vectors $\varphi(k|t) \in \mathbb{R}^{n_u m}$, $k \in [t+1, t+N]$ as:

$$\varphi(k|t) = \begin{cases} [u(k-m)^T, \ldots, u(k|t)^T] & \text{if } k < t+m \\ [u(k-m|t)^T, \ldots, u(k|t)^T] & \text{otherwise.} \end{cases} \tag{18}$$

In addition, we define the current prediction error $\hat{d}(t) \in \mathbb{R}^{n_y}$ as the difference between the measured plant output and the one predicted by the nominal model at time step t:

$$\hat{d}(t) \doteq \tilde{y}(t) - H_c(t)\varphi(t). \tag{19}$$

Then, we consider the following cost function:

$$J(U, \tilde{y}(t), \varphi(t)) \doteq \sum_{k=t}^{t+N-1} (\hat{y}(k+1|t) - y_{\text{des}}(k+1|t))^T Q(\hat{y}(k+1|t) - y_{\text{des}}(k+1|t)) + u(k|t)^T S u(k|t) + \Delta u(k|t)^T R \Delta u(k|t), \tag{20}$$

where:

$$\hat{y}(k+1|t) = H_c(t)\varphi(k+1|t) + \hat{d}(t). \tag{21}$$

In (20), $\tilde{y}(t)$ and $\varphi(t)$ are known parameters and $y_{\text{des}}(k|t)$, $k \in [t+1, t+N]$, are the predicted values of the desired output.

Satisfaction of input constraints can be enforced by the following set of inequalities:

$$\begin{array}{rl} Cu(k|t) \leq & g \\ L\Delta u(k|t) \leq & f \end{array} \quad \forall k \in [t, t+N-1]. \tag{22}$$

In order to define the output constraints, we first introduce the notion of the predicted feasible parameter set, $\mathcal{F}(k|t)$, $k \in [t+1, t+N]$:

$$\mathcal{F}(k|t) = \left\{ H \in \mathbb{R}^{n_y \times n_u m} : A_j(k|t) H_j \leq b_j(k|t) \right\}, \tag{23}$$

where the predicted matrices $A_j(k|t)$ and the vectors $b_j(k|t)$, for $k \in [t+1, t+N-1]$ and $j = 1, \ldots, n_y$ are given as:

$$A(k+1|t) = \begin{cases} A(k|t) & \text{if } r_j(k|t) \leq M' \\ \begin{bmatrix} a_{j1}(k|t) \\ \vdots \\ a_{jr_{j0}}(k|t) \\ a_{jr_{j0}+3}(k|t) \\ \vdots \\ a_{jr_j(t)}(k|t) \end{bmatrix} & \text{otherwise.} \end{cases} \tag{24}$$

$$b(k+1|t) = \begin{cases} b(k|t) + \begin{bmatrix} \mathbf{0}_{r_{jo}} \\ -\underline{\vartheta}_j\left(t - \frac{r_j(t) - r_{j0}}{2}\right) \\ \overline{\vartheta}_j\left(t - \frac{r_j(t) - r_{j0}}{2}\right) \\ \vdots \\ -\underline{\vartheta}_j(t) \\ \overline{\vartheta}_j(t) \end{bmatrix} & \text{if } r_j(k|t) \leq M' \\ \begin{bmatrix} b_{j1}(k|t) \\ \vdots \\ b_{jr_{j0}}(k|t) \\ b_{jr_{j0}+3}(k|t) - \underline{\vartheta}_j\left(k - \frac{r_j(t) - r_{j0}}{2}\right) \\ b_{jr_{j0}+4}(k|t) + \overline{\vartheta}_j\left(k - \frac{r_j(t) - r_{j0}}{2}\right) \\ \vdots \\ b_{jr_j(t)-1}(k|t) - \underline{\vartheta}_j(t) \\ b_{jr_j(t)}(k|t) + \overline{\vartheta}_j(t) \end{bmatrix} & \text{otherwise.} \end{cases} \tag{25}$$

where $a_{ji}(k|t)$ and $b_{ji}(k|t)$ denote the i^{th} row of the matrix $A_j(k|t)$ and the vector $b_j(k|t)$ respectively, $r_j(k|t) = r_j(t) + 2(k-t)$ represents the predicted dimension of the matrices $A_j(k)$ and the vectors $b_j(k)$ that would be obtained by using Algorithm 1 if no rows would be removed (i.e. if the dimension of the matrices and vectors would be allowed to grow without limit in the future), and $M' = M + r_{j0}$ is a constant, with r_{j0} being the constant given by the definition of the set Ω (see Assumption 3).

The matrices that form the predicted feasible parameter sets $\mathcal{F}(k|t), k \in [t+1, t+N-1]$ are formed as if the recursive identification Algorithm 1 would be applied at each time step in the future, but without taking into account the future output measurements, which are unknown at the current time step (i.e. only the inflating effect due to the time-varying dynamics is considered).

The initial predicted matrices $A_j(t|t)$ and the vectors $b_j(t|t), j = 1, \ldots, n_y$ correspond to their actual values at time step t:

$$A_j(t|t) = A_j(t), \; b_j(t|t) = b_j(t). \tag{26}$$

The matrices that form the terminal predicted feasible parameter set are defined just by inequalities that form the uncertainty set Ω to which the model parameter matrix is guaranteed to belong $\forall t \geq 0$ (see Assumption 3):

$$A_j(t+N|t) = A_{j0}, \; b_j(t+N|t) = b_{j0}. \tag{27}$$

The robust satisfaction of the output constraints is then guaranteed by enforcing them for all the parameters inside the predicted feasible parameter sets $\mathcal{F}(k|t), k \in [t+1, t+N]$ and for all disturbance realizations:

$$EH\varphi(k|t) + \overline{d} \leq o, \; \forall H \in \mathcal{F}(t), \; \forall k \in [t+1, t+N], \tag{28}$$

where $\overline{d} = [\overline{d}_1, \ldots, \overline{d}_{n_o}]^T$, and $\overline{d}_l \in \mathbb{R}$, $l = 1, \ldots, n_o$ are given as:

$$\overline{d}_l = \sum_{j=1}^{n_y} |e_{lj}| \epsilon_{d_j},$$

where e_{lj} stands for the element of the l^{th} row and j^{th} column of the matrix E.

Finally, in order to recursively satisfy the input and output constraints (see e.g. Theorem 4.1 in [15]), we introduce an additional constraint on the terminal stage:

$$\varphi(t+N|t) = \varphi(t+N-1|t). \tag{29}$$

This means that we require the terminal regressor to correspond to a steady state, in a way similar to the approach adopted e.g. in [7].

For fixed values of N, Q, S and R, we can now define the finite horizon optimal control problem (FHOCP) to be solved at each time step t:

$$\begin{array}{c} \min\limits_{U, \Lambda} J(U, \tilde{y}(t), \varphi(t)) \\ \text{Subject to: (22), (28), (29),} \end{array} \tag{30}$$

Optimization problem (30) can be reformulated into a quadratic program (QP) of moderate size (see e.g. Lemma 3.2 in [16]). Therefore, the proposed algorithm is well suited for on-line application with processes that have slow dynamics and for which the computational efficiency of the control algorithm is not critical.

IV. Simulation Study

The proposed adaptive control algorithm has been tested in simulation on a three tank system. Fig. 1 shows the system layout. This process consists of three water tanks that are connected in series with narrow pipes attached at the bottom of the tanks and whose cross section can be controlled by valves. Water can be directly pumped from a water reservoir into the two outer tanks, but not into the tank in the middle. One of the outer tanks has a small opening at the bottom through which the water is allowed to leak out into the water reservoir. We assume that all three tanks have the same cross section S. In addition, we assume that the cross sections of the connections between the tanks and of the water outlet have area equal to $\gamma_i S_c, i = 1, 2, 3$, where S_c is a constant term and the gains γ_i are defined by the positions of the corresponding valves. We further denote the water levels in the three tanks with $h_i, i = 1, 2, 3$ and the water flows entering the tanks 1 and 3 with q_1 and q_2.

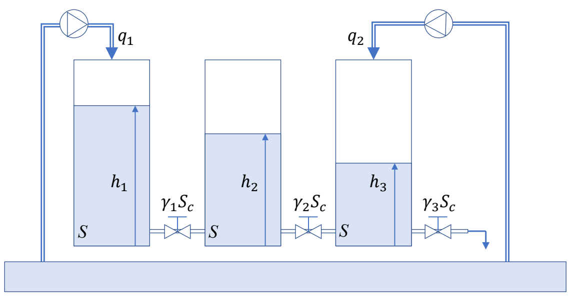

Fig. 1. The three tank system

If we denote the gravity acceleration constant with g, then the dynamic equations that describe the evolution of the water levels in the three tanks are:

$$\begin{aligned}
\frac{dh_1}{dt} &= \frac{q_1 - \gamma_1 S_c sgn(h_1 - h_2)\sqrt{2g(h_1 - h_2)}}{S} \\
\frac{dh_2}{dt} &= \frac{\gamma_1 S_c sgn(h_1 - h_2)\sqrt{2g(h_1 - h_2)} - \gamma_2 S_c sgn(h_2 - h_3)\sqrt{2g(h_2 - h_2)}}{S} \\
\frac{dh_3}{dt} &= \frac{q_2 - \gamma_2 S_c sgn(h_2 - h_3)\sqrt{2g(h_2 - h_2)} - \gamma_3 S_c \sqrt{2gh_3}}{S}
\end{aligned} \tag{31}$$

In simulation, we modify the values of the parameters γ_1 and γ_3 over time, while keeping the value of the parameter γ_2 fixed. Numerical values for the model parameters are listed in Table I.

TABLE I

Numerical values of the three tank model parameters.

S $[cm^2]$	S_c $[cm^2]$	γ_1	γ_2	γ_3
375	3.42	0.5-0.8	0.5	0.75-1.15

We regulate the tank water levels around a steady state defined by $h_1 = 8$ cm, $h_2 = 7$ cm and $h_3 = 6$ cm. Therefore, the simulations are carried out with the linearization of the system (31) around these steady state values, where the plant outputs are the differences of the tank water levels and the steady state levels and the control inputs are the differences of the two water flows with respect to the steady state water flows. The system is controlled with a sampling time of $0.16\ s$.

The described plant has 2 inputs and 3 outputs (i.e. $n_u = 2$ and $n_y = 3$). In the controller, we employ a model given by impulse responses with 12 coefficients to describe the influence of each input to each output (i.e. $m = 12$). The control objective is to regulate the system such that the water level in the tank 2 (i.e. h_2) follows a given reference profile and at the same time satisfy the input and output constraints. The constraints are selected such that the rate and amplitude of both control inputs are limited, that the water level of the first tank stays below 12 cm, that the level of the second tank remains below the level of the first tank and the level of the third tank remains below the level of the second tank and finally that the level of the third tank remains above 0 cm. These input and output constraints yeald the following values for the matrices and vectors in (9):

$$C = L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad g = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}, f = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} 5 \\ 6 \\ 1 \\ 1 \end{bmatrix}.$$

The initial feasible parameter set $\mathcal{F}_0$ and the set of constrains on the model parameter's rate of change $\mathcal{D}$ (see (6)) have been defined by introducing identical box constraints on the impulse response coefficients for each input-output pair. The physics of the considered plant defines the lower bound on each of the impulse response coefficients to be zero, and other bounds are suitably selected such that a large initial uncertainty set $\mathcal{F}_0$ and comparably smaller set $\mathcal{D}$ are obtained. Numerical values of other tuning parameters of the proposed adaptive MPC controller are listed in Table II. In simulations, additive noise uniformly distributed in the range defined by the bounds in Table II was used.

TABLE II

Numerical values of the controller tuning parameters.

ϵ_d	ϵ_v	Q	R	S	N	M
$\begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	22	100

The resulting tank water levels are shown in Fig. 2. In addition to the resulting plant outputs, Fig. 2 also shows the upper and the lower bounds for each of the three outputs, computed considering all possible elements in the feasible parameter set at each time step. As it can be seen, the output constraints are robustly enforced for the whole range of uncertainty.

In order to illustrate the effectiveness of the proposed adaptive control scheme, we compared its performance with the performance of the identical MPC controller that uses least squares with forgetting to cope with the time-varying system behavior. For the simulations, a forgetting factor of 0.9 was used. Both controllers used the same initial guess

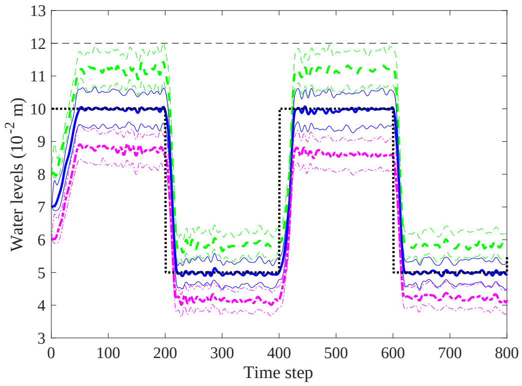

Fig. 2. Resulting tank water levels obtained when the proposed adaptive MPC algorithm is used (thick lines) for the first (green dashed), second (blue solid) and the third (magenta dash-dot) tank, compared with the reference for the water level in the second tank (thick dotted line). In addition to the simulated tank water levels, the uncertainty intervals calculated with respect to the feasible parameter set are also show (thin lines), as well as the constraint of $12\,10^{-2}$ m (black dashed line).

for the plant and the controller that uses least squares has soft enforcement of the output constraints as there are no theoretic guarantees of recursive feasibility. The tank water levels obtained with this controller are shown in Fig. 3. As it can be seen, the use of this controller results in an output constraint violation during simulation. The certainty equivalence adaptive controller with least squares is more aggressive as it does not take the uncertainty into account. On the other hand, although more cautious, the proposed adaptive MPC algorithm for time varying systems is capable of satisfying output constraints and guarantees recursive feasibility.

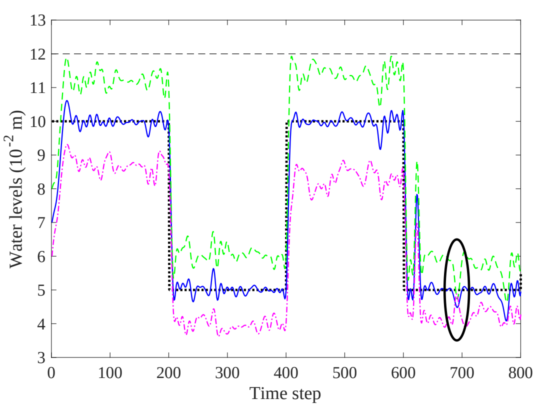

Fig. 3. Resulting tank water levels obtained when the adaptive MPC algorithm that is based on recursive least squares with forgetting is used for the first (green dashed line), second (blue solid) and the third (magenta dash-dot) tank compared with the reference signal for the water level in the second tank (thick dotted line). An example of output constraint violation is marked with a black ellipsoid.

V. Conclusion

This paper proposes a novel adaptive MPC algorithm for handling constrained, linear, time varying systems. It relies on a recursive set membership identification algorithm to keep track of the set of all possible model parameters that is guaranteed to contain the actual plant parameters at each time step, also considering the worst-case parameter changes over time. The MPC formulation results in robust satisfaction of output constraints and recursive feasibility of the finite horizon optimal problem. The effectiveness of the proposed algorithm as been demonstrated in simulation on a tree tank system.

References

[1] V. Adetola, D. DeHan, and M. Guay. Robust adaptive MPC for constrained uncertain nonlinear systems. *International Journal of Adaptive Control and Signal Processing*, 25:155–168, 2011.

[2] A. Aswani, H. Gonzalez, S. Sastry, and C. Tomlin. Provably safe and robust learning-based model predictive control. *Automatica*, 49:1216–1226, 2013.

[3] J. M. Bravo, T. Alamo, and E.F. Camacho. Bounded error identification of systems with time-varying parameters. *IEEE Trans. on Automatic Control*, 51(7):1144 –1150, 2006.

[4] S. Di Cairano. Indirect adaptive model predictive control for linear systems with polytopic uncertainty. In *American Control Conference (ACC), 2016*, pages 3570–3575, Boston, MA, USA, 2016.

[5] M. Canale, L. Fagiano, and M.C. Signorile. Nonlinear model predictive control from data: a set membership approach. *International Journal of Robust and Nonlinear Control*, 24(1):123–139, 2014.

[6] L. Fagiano, G. Schildbach, M. Tanaskovic, and M. Morari. Scenario and adaptive model predictive control of uncertain systems. In *5th IFAC Conference on Nonlinear Model Predictive Control (NMPC)*, pages 352–359, Seville, Spain, 2015.

[7] L. Fagiano and A. Teel. Generalized terminal state constraint for model predictive control. *Automatica*, 49:2622–2631, 2012.

[8] T.A. N. Heirung, E. Ydstie, and B. Foss. Dual adaptive model predictive control. *Automatica*, 80:340–348, 2017.

[9] C. A. Larsson, C. R. Rojas, X. Bombois, and H. Hjalmarsson. Experimental evaluation of model predictive control with excitation (MPC-X) on an industrial depropanizer. *Journal of Process Control*, 31:1–16, 2015.

[10] D. Limon, J. Calliess, and J.M. Maciejowski. Learning-based nonlinear model predictive control. In *IFAC World Congress 2017*, pages 8035 – 8042, Toulouse, France, 2017.

[11] M. Lorenzen, F. Allgöwer, and M. Cannon. Adaptive model predictive control with robust constraint satisfaction. In *IFAC World Congress 2017*, pages 3313 – 3318, Toulouse, France, 2017.

[12] G. Marafioti, R. Bitmead, and M. Hovd. Persistently exciting model predictive control. *Adaptive Control and Signal Processing*, 28:536–552, 2014.

[13] D.Q. Mayne. Model predictive control: Recent developments and future promise. *Automatica*, 50:2967–2986, 2014.

[14] S.J. Qin and T.A. Badgwell. A survey of industrial model predictive control technology. *Control Engineering Practice*, 11:733–764, 2003.

[15] M. Tanaskovic, L. Fagiano, and V. Gligorovski. Adaptive model predictive control for constrained, linear time varying systems. *arXiv:1712.07548 (available online)*, 2017.

[16] M. Tanaskovic, L. Fagiano, R. Smith, and M. Morari. Adaptive receding horizon control for constrained mimo systems. *Automatica*, 50:3019–3029, 2014.

[17] M. Tanaskovic, D. Sturzenegger, R. Smith, and M. Morari. Robust adaptive model predictive building climate control. *IFAC World Congress*, 2017.

[18] X. Wang, Y. Sun, and K. Deng. Adaptive model predictive control of uncertain constrained systems. In *Proceedings of the 2014 American Control Conference*, pages 2857 – 2862, Portland, OR, 2014.

[19] J. Zhou, S. Di Cairano, and C. Danielson. Indirect adaptive mpc for output tracking of uncertain linear polytopic systems. In *Proceedings of the 2017 American Control Conference*, pages 3054 – 3059, Seattle, WA, 2017.