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Robust predictive control with data-based multi-step prediction models

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***Abstract*— In this paper a novel method for the design of MPC controllers based on multi-steps ahead external representation system models is proposed. These models are assumed to be identified from available data, e.g., using the set-membership approach, that computes a guaranteed estimate of the model uncertainty bounds. Exploiting this information, the proposed algorithm guarantees input and output constraint satisfaction, recursive feasibility, and robust convergence properties. A simulation case study is shown to demonstrate the effectiveness of the proposed approach.**

I. Introduction

The idea to jointly use identification and control design tools is old-standing, see e.g. [15] and [20], and is motivated by the need to reduce the time and effort required by the modeling phase, which could be cumbersome or even impossible if based on physical models. This need becomes particularly dramatic nowadays, since more and more complex, large-scale, and heterogeneous systems are required to be addressed [18], [2], [19], [7], [9], preventing modelling approaches purely based on physical equations to be effective. On the other hand, we are now witnessing considerable technological progresses in electronics, communications (e.g., with sensor networks [3]), and computer science, giving us the opportunity of transmitting, collecting, storing, and managing huge quantities of data, extracted from the systems and plants to be controlled.

In this framework, a very wide literature is nowadays available on the so-called identification for control approach referred to linear systems, see for instance [6], [5], or on the use of nonlinear models based on neural networks, see for example [16]. In the control design phase, most of the proposed approaches used so far rely on the so-called *certainty equivalence* principle: uncertainties in the model description are neglected and the nominal model is used. To overcome the intrinsic limitations of this approach, different strategies can be adopted. Among them, stochastic and deterministic (i.e., where the disturbance is unknown but assumed to be bounded in a known set) robust control methods have been proposed. In the first case, the inherent nondeterminism of the model is explicitly considered, while in the second case a worst-case approach is adopted. For example, a wide literature is currently devoted to robust stochastic (see the recent survey [4]) and deterministic (e.g., the popular robust "tube-based" approach described in [13]) algorithms based on Model Predictive Control (MPC), which is nowadays the largely most popular control design method.

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Motivated by the above considerations, by the recent advances in Machine Learning techniques, and by the ever increasing computing power, the research on the combined use of learning techniques and MPC has recently received an increasing interest. Two recent notable contributions in this field have been given in [1] and [10]. In the first paper, linear models with bounded uncertainties are considered and deterministic guarantees on robustness and safety are provided. In the second paper, [10], a nonparametric machine learning algorithm is used together with a robust MPC technique for control of systems described by nonlinear models.

In this research stream this paper proposes a new algorithm for MPC control of linear systems described by a multi-steps ahead external representation. The potential advantages of using different models for prediction, one for each future prediction step, has been previously highlighted in [20], [8], and [12], where however the control design required the solution to a nonlinear optimization problem. In the approach described in the following the multistep predictors are assumed to be built independently from each other according to the algorithm described in [21], which relies on the Set Membership theory [22], [14] . As shown in [21], the adopted learning approach is based on the solution to LP and QP problems, so that its computational load is limited, and allows for the computation of different worst case prediction errors, one for each prediction step, so avoiding the classical accumulation of prediction errors raising from iteration of one-step models. Moreover, noise on the output measures is allowed.

Given the identified set of models and their related worst-case errors, the proposed robust MPC algorithm is based on the tube-based approach, suitably modified to cope with the peculiarities of the adopted model. The fundamental properties of recursive feasibility and convergence are proven and a simulation example is reported to witness the performance of the method.

The paper is organized as follows: in Section II the predictive control problem is stated and multi-steps ahead external representation system models are introduced. The complete MPC algorithm is described in Section III, where the proofs of recursive feasibility and convergence are also reported. Section IV reports a benchmark simulation example taken from the literature, see [20], that witnesses the effectiveness of the proposed approach. Finally, Section V summarizes the main results of this paper and presents future work perspectives.

II. Statement of the problem

In this paper we consider a SISO system with input variable $u(k) \in \mathbb{R}$ and measured output $y(k) \in \mathbb{R}$, $k \in \mathbb{N}$.
We assume that, from the application of a suitable identification/learning method, the following external representation system models, describing the evolution of the measured output variable $y(k)$ are available, for all $p = 1,\ldots,\overline{p}, \overline{p} \in \mathbb{N}, \overline{p} \geq 1$.

$$\begin{aligned} y(k+p) \quad = \quad & \theta^T_{p,Y_o} \begin{bmatrix} y(k) \\ \vdots \\ y(k-o+1) \end{bmatrix} + \theta^T_{p,U_o} \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-o+1) \end{bmatrix} \\ & + \theta^T_{p,U_p} \begin{bmatrix} u(k) \\ \vdots \\ u(k+p-1) \end{bmatrix} + w_p(k) \end{aligned} \tag{1}$$

where θ_{p,Y_o}, $\theta_{p,U_o}, \theta_{p,U_p}$ are vector of known parameters, and where $w_p(k)$ is an unknown, but bounded, disturbance term, i.e., there exist, for all $p = 1,\ldots,\overline{p}$, a bounded and convex set $\mathbb{W}_p$ containing the origin in its interior such that $w_p(k) \in \mathbb{W}_p$ for all $k \geq 0$. Note that the magnitude of set $\mathbb{W}_p$ depends on a number of factors, including the number of data considered in the identification procedure, the considered class of models, and the noise acting on the system (e.g., the measurement noise). Therefore, if necessary, the magnitude of $\mathbb{W}_p$ can be reduced by a more accurate choice of the model class and by considering more data in the learning phase. However, a lower bound is to be expected, in view of the fact that the data (and possibly the dynamic system under control) are in general affected by noise and perturbations.

Equation (1) represents how the output $y(k)$ evolves on a p-steps ahead basis. The use of these models is assumed to be beneficial in a predictive control context in view of the fact that the uncertainty associated to (1) is presumably smaller than the uncertainty resulting from iterating p times the 1-step evolution model (i.e., corresponding with (1) where p is set to 1), and therefore the optimization-based predictive controller devised based on them results consequently more precise and effective. In particular, this property is guaranteed if the model (1) has been derived using set membership identification methods as described in [21].

The scope of this paper is to devise a theoretically sound robust predictive optimization-based control method for regulation that is able to optimally take advantage of the availability of the models (1) for computing the forecasts during the whole prediction horizon. The controller must be able to guarantee the fulfillment of the following input and output constraints, for all $k \geq 0$.

$$u(k) \in \mathbb{U} \tag{2a}$$

$$y(k) \in \mathbb{Y} \tag{2b}$$

where $\mathbb{U}$ and $\mathbb{Y}$ are suitable convex sets containing the origin in their interior. Note that, although here the system is defined in a SISO context, the extension to the multi-input and multi-output framework is straightforward.

III. Formulation of the MPC problem

A. Derivation of the state space representation and system assumptions

A state space representation of the system evolution is required, in order to formulate in a sound way the corresponding robust MPC problem. To do so we define $X(k) = (y(k),\ldots,y(k-o+1),u(k-1),\ldots,u(k-o+1))^T \in \mathbb{R}^{2o-1}$ and $U(k) = (u(k),\ldots,u(k+\overline{p}))^T \in \mathbb{R}^{\overline{p}+1}$ as the state and the input, respectively, variables. Thanks to this, the following possibly non-minimal representation of the 1-step evolution model is derived from (1) with $p = 1$.

$$\begin{aligned} X(k+1) &= AX(k) + BU(k) + M_1 w_1(k) \\ &= AX(k) + B_1 u(k) + M_1 w_1(k) \end{aligned} \tag{3}$$

where

$$A = \begin{bmatrix} \theta^T_{1,Y_0} & \theta^T_{1,U_0} \\ I_{o-1} \quad 0_{o-1,1} & 0_{o-1,o-1} \\ 0_{o-1,o} & 0_{1,o-1} \\ & I_{o-2} \quad 0_{o-2,1} \end{bmatrix}$$

$$B = \begin{bmatrix} \theta^T_{1,U_1} 0_{1,\overline{p}} \\ 0_{o-1,\overline{p}+1} \\ 1 \quad 0_{1,\overline{p}} \\ 0_{o-2,\overline{p}+1} \end{bmatrix}, B_1 = \begin{bmatrix} \theta^T_{1,U_1} \\ 0_{o-1,1} \\ 1 \\ 0_{o-2,1} \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 \\ 0_{2(o-1),1} \end{bmatrix}$$

For $p = 1,\ldots,\overline{p}$, we rewrite the p-steps ahead evolution models (1) as system output equations as follows.

$$y(k+p) = C_p X(k) + D_p U(k) + w_p(k) \tag{4}$$

where $C_p = \begin{bmatrix} \theta^T_{p,Y_0} & \theta^T_{p,U_0} \end{bmatrix}$, $D_p = \begin{bmatrix} \theta^T_{p,U_p} & 0_{1,\overline{p}+1-p} \end{bmatrix}$. Let us introduce matrices $C_0 = [1,0,0,\ldots,0] \in \mathbb{R}^{1\times 2o-1}$ and $D_0 = 0_{1,\overline{p}+1}$ such that we can write $y(k) = C_0 X(k) + D_0 U(k)$.
The following assumption will be considered in the remainder of the paper.

Assumption 1: The pair (A, B_1) is stabilizable. □

B. Structure of the controller

In this paper an MPC controller is devised that, at each time instant k, uses the p-steps ahead models (1) to predict in the best possible way the future evolution of the output variable. Since models (3) and (1) are affected by bounded disturbances, a tube-based robust control method is used, inspired by the algorithm proposed in [13]. As a result, the input $u(k)$, to be applied to the system at time instant k, is defined as the sum of two components as follows.

$$u(k) = \hat{u}(k|k) + K(X(k) - \hat{X}(k|k)) \tag{5}$$

The component $\hat{u}(k|k)$ is computed as the result to a suitable optimization problem, to be defined based on the nominal (i.e., unperturbed) prediction models derived from (1), while the second component (i.e., $K(X(k) - \hat{X}(k|k))$) is defined using a suitable proportional control law, aiming to reduce the displacement of the state $\hat{X}(k|k)$ of a suitably defined nominal dynamic system with respect to the real data $X(k)$ available at time k.

The nominal dynamic system is defined based on (3), i.e.,

$$\hat{X}(k+1) = A\hat{X}(k) + B_1\hat{u}(k) \tag{6}$$

The difference between the real available data vector $X(k)$ and the state of the nominal system is defined as $e(k) = X(k) - \hat{X}(k)$. From (3) and (6), it evolves according to:

$$e(k+1) = (A + B_1K)e(k) + M_1w_1(k) \tag{7}$$

Let us assume that the gain K is defined in such a way that the closed-loop transition matrix $A + B_1K$ is Schur stable, which is possible thanks to Assumption 1. Let us denote with $\mathbb{E}$ a (minimal, if possible) robust positively invariant (RPI) [17] set for the system (7). Similarly to [13], the constraints and the optimization problem will be defined with reference to the unperturbed model, and specifically to (6). This will require to define suitable tightened state and input constraints, that allow to account for the difference between $\hat{X}(k)$ and $X(k)$.

C. Constraints

Similarly to [13], it is first necessary to constrain $\hat{X}(k)$ at time k to lie in the neighborhood of $X(k)$, i.e

$$X(k) - \hat{X}(k) \in \mathbb{E} \tag{8a}$$

As discussed, the constraints will be defined only with reference to the state-space model (6). Regarding the input variable, to guarantee that (2a) holds at time $k+p$, $p = 0,\dots,\overline{p}$, it is enough to enforce the following tightened constraint, for all $p = 0,\dots,\overline{p}$.

$$\hat{u}(k+p) \in \mathbb{U} \ominus K\mathbb{E} \tag{8b}$$

Regarding the output, to guarantee that (2b) holds at time $k+p$, $p = 0,\dots,\overline{p}$, we enforce the following tightened constraint, for all $p = 0,\dots,\overline{p}$.

$$C_0\hat{X}(k+p) \in \mathbb{Y} \ominus C_0\mathbb{E} \tag{8c}$$

where $\hat{X}(k+p) = A^p\hat{X}(k) + \sum_{j=1}^{p} A^{j-1}B_1\hat{u}(k+p-j)$.

To guarantee recursive feasibility, we also need to enforce a terminal constraint of the type

$$\hat{X}(k+\overline{p}+1) \in \mathbb{X}_F \tag{8d}$$

where $\mathbb{X}_F$ is defined as a positively invariant set for the system $\hat{X}(k+1) = (A + B_1K)\hat{X}(k)$ that verifies $C_0\mathbb{X}_F \subseteq \mathbb{Y} \ominus C_0\mathbb{E}$ and $K\mathbb{X}_F \subseteq \mathbb{U} \ominus K\mathbb{E}$. For consistency, the following assumption is required.

Assumption 2: There exists a ball $\mathscr{B}$ in space $\mathbb{R}$, centered at the origin and with radius ε, such that

$$C_0\mathbb{E} \oplus \mathscr{B} \subseteq \mathbb{Y} \tag{9a}$$

$$K\mathbb{E} \oplus \mathscr{B} \subseteq \mathbb{U} \tag{9b}$$

□

Notably, Assumption 2 can be regarded as a minimal requirement with respect to the uncertainty associated to the identification procedure. In fact, the magnitude of set $\mathbb{E}$ is directly proportional to the magnitude of set $\mathbb{W}_1$ which, as discussed in Section II, represents the uncertainty resulting from the identification phase. In case Assumption 2 is not satisfied it will be necessary to step back to the identification stage and, if possible, improve it (i.e., reduce the related uncertainty level). As discussed, this can be done either by increasing the number of data or even considering a different class of models.

D. Cost function

The p steps ahead output predictor corresponding to (4) is computed using the following equation.

$$\hat{y}_p(k) = C_p\hat{X}(k) + D_p\hat{U}(k) \tag{10}$$

where $\hat{U}(k) = (\hat{u}(k),\dots,\hat{u}(k+\overline{p}))$.

The cost function to be minimized at time step k is

$$J(k) = \sum_{p=0}^{\overline{p}} \|\hat{y}_p(k)\|^2_{Q_p} + \|\hat{u}(k+p)\|^2_{R_p} + \|\hat{X}(k+\overline{p}+1)\|^2_P \tag{11}$$

The terminal state value $\hat{X}(k+\overline{p}+1)$ is obtained by iterating the state equation (3) $\overline{p}+1$ times, i.e.,

$$\hat{X}(k+\overline{p}+1) = A^{\overline{p}+1}\hat{X}(k) + \Gamma\hat{U}(k) \tag{12}$$

where

$$\Gamma = \begin{bmatrix} A^{\overline{p}}B_1 & \dots & B_1 \end{bmatrix}$$

For the definition of the weights Q_p, R_p, and P we must preliminarily define the following matrices.

$$\Psi = \begin{bmatrix} C_0A & C_0B + D_0H_1 \\ \vdots & \vdots \\ C_{\overline{p}}A & C_{\overline{p}}B + D_{\overline{p}}H_1 \end{bmatrix}, \bar{\Psi} = \begin{bmatrix} C_1 & D_1 \\ \vdots & \vdots \\ C_{\overline{p}} & D_{\overline{p}} \\ A^{\overline{p}+1} & \Gamma \\ 0 & I_{\overline{p}+1} \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0_{\overline{p},1} & I_{\overline{p}} \\ 0 & 0_{1,\overline{p}} \end{bmatrix} \tag{13}$$

Also, we write $\mathscr{Q}$ =diag$(Q_0,\dots,Q_{\overline{p}})$, and $\bar{\mathscr{Q}}$ =diag$(Q_1,\dots,Q_{\overline{p}},T_N,\mathscr{R})$, where T_N is a positive definite matrix to be used as a further tuning knob and $\mathscr{R}$ =diag$(R_0/2, R_1 - R_0,\dots,R_{\overline{p}} - R_{\overline{p}-1})$. We will assume that the following conditions are verified:

$$(A + B_1K)^TP(A + B_1K) - P = -T_N - K^TR_{\overline{p}}K \tag{14a}$$

$$\Psi^T\mathscr{Q}\Psi \leq \bar{\Psi}^T\bar{\mathscr{Q}}\bar{\Psi} \tag{14b}$$

$$T_N > 0, \quad \mathscr{R} > 0, \quad \mathscr{Q} > 0 \tag{14c}$$

It is worth noting that condition (14a) is a standard Lyapunov equation with respect to the stable transition matrix $(A + B_1K)$. Overall, the set of inequalities (14) can be cast as LMIs and solved using standard tools, e.g., YALMIP [11].

E. The optimization problem and main result

The optimization problem, to be solved at each time instant $k \geq 0$, reads

$$J(k|k) = \min_{\hat{X}(k),\hat{U}(k)} J(k) \tag{15}$$

subject to (8).

If admissible, the solution to the optimization problem (15) is denoted $\hat{X}(k|k), \hat{U}(k|k) = (\hat{u}(k|k), \ldots, \hat{u}(k+\overline{p}|k))$, and $u(k)$ in (5) is implemented on the system according to the Receding Horizon principle. Also, we denote with the notation $\hat{X}(k+p|k)$ the future nominal state predictions generated using (6) with input $\hat{U}(k|k) = (\hat{u}(k|k), \ldots, \hat{u}(k+\overline{p}|k))$. The following results can be proved.

Theorem 1: If the optimization problem is feasible at time step $k=0$ then it is feasible at all time steps $k>0$. Also, for all $k \geq 0$, the constraints (2) are satisfied and $\hat{y}_0(k) \to 0$ as $k \to \infty$. Finally, $d(y(k), C_0\mathbb{E}) \to 0$ as $k \to \infty$, where $d(\alpha, \beta)$ denotes the distance from point α to set β. □

Proof: See the Appendix. ■

IV. Numerical results

The proposed approach has been tested on the benchmark example previously considered in [8], [20] and [12]. More specifically, we consider the data generated according to the continuous-time system

$$z(t) = \frac{458}{(s+1)(s^2+30s+229)} u(t)$$

The output measurements are collected with a sampling time $T_s = 0.2$ s according to the output equation

$$y(k) = z(kT_s) + d(kT_s)$$

where $d(t)$ is a bounded noise such that, for all t, $d(t) \in [-0.2, 0.2]$.

The identification procedure considers a data set composed of $N = 500$ input-output data samples. The used identification algorithm, discussed in [21], allows to identify the discrete-time models (1) for all $p = 1, \ldots, \overline{p} = 7$, where the order is $o = 2$. Note, in passing, that the real system, of order 3, does not belong to the selected model class of order 2, and this makes the simulation tests more realistic, however at the price of a bigger uncertainty resulting from the identification phase.

The algorithm proposed in this paper is used to regulate the output $y(t)$ to zero, starting from a feasible initial condition and applying (5), where the gain K is computed by means of a LQ auxiliary control law and the tuning of the MPC regulator satisfies (14). Constraints are defined such that the sets for $y(k)$ and $u(k)$ are $\mathbb{Y} = [-10, 10]$ and $\mathbb{U} = [-3, 3]$ respectively, then tightened according to (16c) and (16b) in the optimization problem (15). By definition of the algorithm it is necessary to collect at least o samples in order to properly define the required state $X(k) = (y(k), \ldots, y(k-o+1), u(k-1), \ldots, u(k-o+1))^T \in \mathbb{R}^{2o-1}$. To initialize the input variable, we set $u(k) = 0$ for $k = 1, \ldots, 4$, i.e., the system is in open loop with null input up to time instant 1 s. An alternative choice could be to use a pre-stabilizing controller.

To better show the variability of the solution for different disturbance signals, 20 runs are simulated with different realizations of $d(t)$.

Figure 1 shows the input $\hat{u}(k|k)$ resulting from the optimizer. Note that the constraints (16b) (represented with a dashed black line) is active at time $t = 1$ s; as expected $\hat{u}(k|k) \to 0$ as $k \to +\infty$ for each disturbance realization.

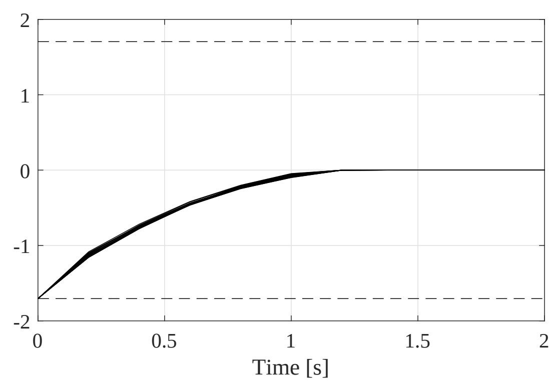

Fig. 1. Trajectories of $\hat{u}(k|k)$ for any disturbance realizations. Solid lines: $\hat{u}(k|k)$; dashed black lines: tightened constraints (16b) applied on $\hat{u}(k|k)$.

The corresponding trajectories of $u(k)$ are shown in Figure 2. Note that, in view of the presence of the disturbances, $u(k)$ does not asymptotically tend to zero but to a bounded set included in $K\mathbb{E}$, in view of (5), (16a), and of the fact that $\hat{u}(k|k)$ and $\hat{X}(k|k)$ tend to zero.

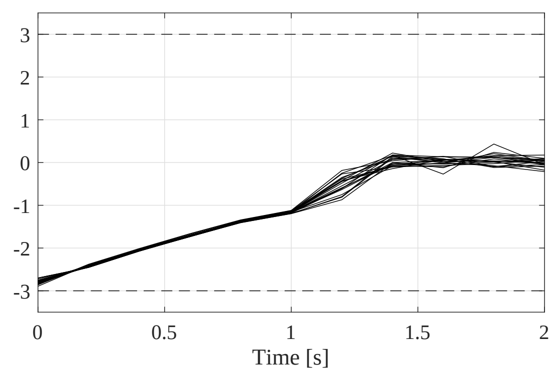

Fig. 2. Trajectories of $u(k)$ for any disturbance realizations. Solid lines: $u(k)$; dashed black lines: constraints (2a).

Figure 3 shows the output trajectory, starting from a feasible initial condition. Consistently with Theorem 1, the output does not asymptotically converge to zero but, for all the disturbance realizations, to the bounded set $C_0\mathbb{E}$, whose bounds are indicated in Figure 3 by the black dotted lines.

Finally, Figure 4 shows the nominal output obtained as $\hat{y}(k|k) = C_0\hat{X}(k|k)$, $\hat{X}(k|k)$ resulting from the optimization (15). As expected, this signal converges to zero asymptotically

V. Conclusions

In this paper a novel method for the design of MPC controllers with data-based multi-steps ahead external representation system models has been described. The use of these models is motivated by the predictive control scheme adopted in the paper. Under the condition that the models are affected by a bounded and known uncertainty, the proposed algorithm

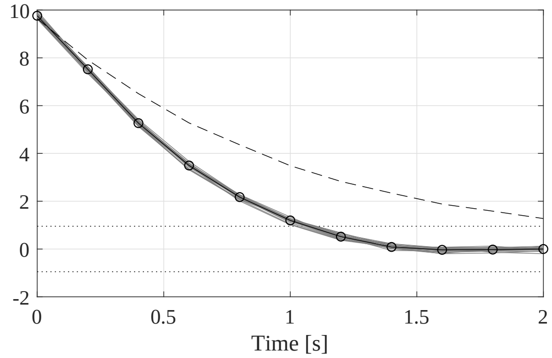

Fig. 3. Trajectories of $y(k)$ for any disturbance realizations and open loop response. Grey lines: $y(k)$; solid line with circles: average trajectory of the realizations; dotted black lines: bounds of set $C_0\mathbb{E}$; dashed line: open loop trajectory.

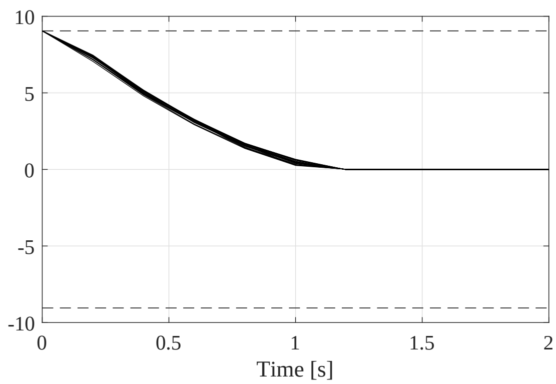

Fig. 4. Trajectories of $\hat{y}(k|k) = C_o\hat{X}(k|k)$ for any disturbance realizations. Solid lines: $\hat{y}(k|k)$; dashed black lines: tightened constraints (16c) applied on $\hat{y}(k|k)$.

guarantees input and output constraint satisfaction, recursive feasibility, and robust convergence properties. A simulation case study is shown to demonstrate the effectiveness of the approach.

This work should be considered as a preliminary step: research will be devoted to a number of issues. For example, the extension to MIMO and non-linear models is envisioned. Also, the control scheme will be extended to address the tracking problem. Another interesting issue, that will be deeply investigated in the future, is the interplay between identification and control: for example, how to steer the identification procedure in order to develop models that are particularly prone to the control phase is a still open issue.

Appendix

Proof of Theorem 1.

The proof of Theorem 1 is here divided in three steps:

- Proof of recursive feasibility of the optimization problem (15).
- Proof that constraints (2) are satisfied.
- Proof of convergence.

Recursive feasibility.

The proof is conducted by induction. Assume that, at instant k, a solution to the optimization problem (15) exists. All constraints (8) are therefore verified by the trajectories $\hat{X}(k+p|k)$ and $\hat{U}(k|k) = (\hat{u}(k|k), \ldots, \hat{u}(k+\overline{p}|k))$, and more specifically

$$X(k) - \hat{X}(k|k) \in \mathbb{E} \tag{16a}$$

$$\hat{u}(k+p|k) \in \mathbb{U} \ominus K\mathbb{E}, \quad p = 0, \ldots, \overline{p} \tag{16b}$$

$$C_0\hat{X}(k+p|k) \in \mathbb{Y} \ominus C_0\mathbb{E}, \quad p = 0, \ldots, \overline{p} \tag{16c}$$

$$\hat{X}(k+\overline{p}+1|k) \in \mathbb{X}_F \tag{16d}$$

Finally, the input $u(k)$ is defined according to (5).

At step $k+1$, $X(k+1) = AX(k) + B_1u(k) + M_1w_1(k)$. We can show that a feasible, although possibly suboptimal, solution to (15) at step $k+1$ can be defined, i.e., as $\hat{X}(k+1|k), \hat{U}(k+1|k) = \big(\hat{u}(k+1|k), \ldots, \hat{u}(k+\overline{p}|k), K\hat{X}(k+\overline{p}+1|k)\big)$. First of all, we compute that $X(k+1) - \hat{X}(k+1|k) = (A + B_1K)(X(k) - \hat{X}(k|k)) + M_1w_1(k) \in \mathbb{E}$ in view of (16a) and of the fact that $\mathbb{E}$ is RPI.

Also, $\hat{u}(k+p|k) \in \mathbb{U} \ominus K\mathbb{E}$ in view of (16b), for all $p = 1, \ldots, \overline{p}$; also, $K\hat{X}(k+\overline{p}+1|k) \in K\mathbb{X}_F \subseteq \mathbb{U} \ominus K\mathbb{E}$ in view of (16d) and of the definition of $\mathbb{X}_F$.

Thirdly, $C_0\hat{X}(k+p|k) \in \mathbb{Y} \ominus C_0\mathbb{E}$ for all $p = 1, \ldots, \overline{p}$ in view of (16c) and $C_0\hat{X}(k+\overline{p}+1|k) \in C_0\mathbb{X}_F \subseteq \mathbb{Y} \ominus C_0\mathbb{E}$ in view of (16d) and of the definition of $\mathbb{X}_F$. Finally, it holds that $\hat{X}(k+\overline{p}+2|k) = (A+B_1K)\hat{X}(k+\overline{p}+1|k) \in \mathbb{X}_F$ in view of (16d) and of the positive invariance of $\mathbb{X}_F$. Since feasibility holds by assumption at time $k = 0$ then, by induction, it is guaranteed also for all $k > 0$.

Constraint satisfaction.

In view of the feasibility of the problem (15) at any time instant $k \geq 0$, it results that constraints (16) are verified. Therefore, from (5), (16a), and (16b), $u(k) = \hat{u}(k+p|k) + K(X(k) - \hat{X}(k|k)) \in (\mathbb{U} \ominus K\mathbb{E}) \oplus K\mathbb{E} = \mathbb{U}$, proving (2a). Also, from (16a) and (16c), $y(k) = C_0X(k) = C_0\hat{X}(k|k) + C_0(X(k) - \hat{X}(k|k)) \in (\mathbb{Y} \ominus C_0\mathbb{E}) \oplus C_0\mathbb{E} = \mathbb{Y}$, proving (2b).

Convergence.

We compute, from (11), that

$$\begin{aligned} J(k|k) \quad &= \textstyle\sum_{p=0}^{\overline{p}} \| \begin{bmatrix} C_p & D_p \end{bmatrix} \begin{bmatrix} \hat{X}(k|k) \\ \hat{U}(k|k) \end{bmatrix} \|_{Q_p}^2 \\ &+ \|\hat{u}(k+p|k)\|_{R_p}^2 + \|\hat{X}(k+\overline{p}+1|k)\|_P^2 \end{aligned} \tag{17}$$

At step $k+1$, in view of optimality the optimal cost function $J(k+1|k+1)$ verifies $J(k+1|k+1) \leq J(k+1|k)$, where $J(k+1|k)$ is the cost function obtained at step $k+1$ if the feasible solution $\hat{X}(k+1|k), \hat{U}(k+1|k)$ is applied. More specifically

$$\begin{aligned} J(k+1|k) = \textstyle\sum_{p=0}^{\overline{p}} \| \begin{bmatrix} C_p & D_p \end{bmatrix} \begin{bmatrix} \hat{X}(k+1|k) \\ \hat{U}(k+1|k) \end{bmatrix} \|_{Q_p}^2 \\ + \|\hat{u}(k+1+p|k)\|_{R_p}^2 + \|\hat{X}(k+\overline{p}+2|k)\|_P^2 \end{aligned} \tag{18}$$

where $\hat{u}(k+\overline{p}+1|k) = K\hat{X}(k+\overline{p}+1|k)$. We compute, from (17) and (18), that $J(k+1|k+1) - J(k|k) \leq J(k+1|k) - J(k|k) \leq -(\|\hat{y}_0(k|k)\|_{Q_0}^2 + \|\hat{u}(k|k)\|_{R_0}^2) + \sum_{p=0}^{\overline{p}-1}(\| \begin{bmatrix} C_p & D_p \end{bmatrix} \begin{bmatrix} \hat{X}(k+1|k) \\ \hat{U}(k+1|k) \end{bmatrix} \|_{Q_p}^2 -$

$\| \begin{bmatrix} C_{p+1} & D_{p+1} \end{bmatrix} \begin{bmatrix} \hat{X}(k|k) \\ \hat{U}(k|k) \end{bmatrix} \|^2_{Q_{p+1}} + \|\hat{u}(k+1+p|k)\|^2_{R_p} - \|\hat{u}(k+1+p|k)\|^2_{R_{p+1}}) + \| \begin{bmatrix} C_{\overline{p}} & D_{\overline{p}} \end{bmatrix} \begin{bmatrix} \hat{X}(k+1|k) \\ \hat{U}(k+1|k) \end{bmatrix} \|^2_{Q_{\overline{p}}} + \|K\hat{X}(k+1+\overline{p}|k)\|^2_{R_{\overline{p}}} - \|\hat{X}(k+1+\overline{p}|k)\|^2_P + \|(A+B_1K\hat{X}(k+1+\overline{p}|k)\|^2_P$.

Also, $\hat{U}(k+1|k) = H_1\hat{U}(k|k) + H_2K_{\text{aux}}\hat{X}(k+\overline{p}+1|k)$, where $\hat{X}(k+\overline{p}+1|k) = A^{\overline{p}+1}\hat{X}(k|k) + \Gamma\hat{U}(k|k)$, being $H_2 = \begin{bmatrix} 0_{1,\overline{p}} & 1 \end{bmatrix}^T$, and so we can write

$$\begin{bmatrix} \hat{X}(k+1|k) \\ \hat{U}(k+1|k) \end{bmatrix} = \begin{bmatrix} A & B \\ H_2K_{\text{aux}}A^{\overline{p}+1} & H_1 + H_2K_{\text{aux}}\Gamma \end{bmatrix} \begin{bmatrix} \hat{X}(k|k) \\ \hat{U}(k|k) \end{bmatrix} \tag{19}$$

For notational simplicity, let

$$\hat{\xi}(k|k) = \begin{bmatrix} \hat{X}(k|k) \\ \hat{U}(k|k) \end{bmatrix}$$

We define for $p \geq 1$, $\Delta R_p = R_p - R_{p-1}$, in such a way that it is possible to write $-\|\hat{u}(k|k)\|^2_{R_0/2} + \sum_{p=0}^{\overline{p}-1} \left(\|\hat{u}(k+1+p|k)\|^2_{-\Delta R_{p+1}} \right) = -\|\hat{U}(k|k)\|^2_{\mathscr{R}} = -\| \begin{bmatrix} 0 & I_{\overline{p}+1} \end{bmatrix} \hat{\xi}(k|k)\|^2_{\mathscr{R}}$.

Recalling that $D_pH_2 = 0$, for all $p = 1, \ldots, \overline{p}$, we can eventually write that $J(k+1|k+1) - J(k|k) \leq -(\|y(k|k)\|^2_{Q_0} + \|\hat{u}(k|k)\|^2_{R_0/2}) + \sum_{p=0}^{\overline{p}-1}(\| \begin{bmatrix} C_pA & C_pB + D_pH_1 \end{bmatrix} \hat{\xi}(k|k)\|^2_{Q_p} - \| \begin{bmatrix} C_{p+1} & D_{p+1} \end{bmatrix} \hat{\xi}(k|k)\|^2_{Q_{p+1}}) - \| \begin{bmatrix} 0 & I_{\overline{p}+1} \end{bmatrix} \hat{\xi}(k|k)\|^2_{\mathscr{R}}$
$+ \| \begin{bmatrix} C_{\overline{p}}A & C_{\overline{p}}B + D_{\overline{p}}H_1 \end{bmatrix} \hat{\xi}(k|k)\|^2_{Q_{\overline{p}}} + \|K\hat{X}(k+\overline{p}+1|k)\|^2_{R_{\overline{p}}} - \|\hat{X}(k+\overline{p}+1|k)\|^2_P + \|(A+B_1K)\hat{X}(k+\overline{p}+1|k)\|^2_P$.

Then, using (14a) we obtain that $J(k+1|k+1) - J(k|k) \leq -(\|\hat{y}_0(k|k)\|^2_{Q_0} + \|\hat{u}(k|k)\|^2_{R_0/2}) + \sum_{p=0}^{\overline{p}-1}(\| \begin{bmatrix} C_pA & C_pB + D_pH_1 \end{bmatrix} \hat{\xi}(k|k)\|^2_{Q_p} - \| \begin{bmatrix} C_{p+1} & D_{p+1} \end{bmatrix} \hat{\xi}(k|k)\|^2_{Q_{p+1}}) - \| \begin{bmatrix} 0 & I_{\overline{p}+1} \end{bmatrix} \hat{\xi}(k|k)\|^2_{\mathscr{R}} + \| \begin{bmatrix} C_{\overline{p}}A & C_{\overline{p}}B + D_{\overline{p}}H_1 \end{bmatrix} \hat{\xi}(k|k)\|^2_{Q_{\overline{p}}} - \| \begin{bmatrix} A^{\overline{p}+1} & \Gamma \end{bmatrix} \hat{\xi}(k|k)\|^2_{T_N}$ and, in a more compact form,

$$J(k+1|k+1) - J(k|k) \leq -\|\hat{y}_0(k|k)\|^2_{Q_0} + \|\hat{\xi}(k|k)\|^2_{\Omega} \tag{20}$$

where $\Omega = \Psi^T \mathscr{Q} \Psi - \bar{\Psi}^T \bar{\mathscr{Q}} \bar{\Psi}$.

Finally, if (14b) is satisfied, then

$$J(k+1|k+1) - J(k|k) \leq -\|\hat{y}(k|k)\|^2_{Q_0} \tag{21}$$

In view of (21), then $\hat{y}_0(k|k) \to 0$ as $k \to +\infty$. Also, recalling (16a), $C_0(X(k) - \hat{X}(k|k)) = y(k) - \hat{y}_0(k|k) \in C_0\mathbb{E}$, for all k which allows to conclude the proof.

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