Generalized terminal state constraint for model predictive control

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Abstract

A terminal state equality constraint for Model Predictive Control (MPC) laws is investigated, where the terminal state/input pair is not fixed a priori but it is a free variable in the optimization. The approach, named "generalized" terminal state constraint, can be used for both tracking MPC (i.e. when the objective is to track a given steady state) and economic MPC (i.e. when the objective is to minimize a cost function which does not necessarily attains its minimum at a steady state). It is shown that the proposed technique provides, in general, a larger feasibility set with respect to existing approaches, given the same prediction horizon. Moreover, a new receding horizon strategy is introduced, exploiting the generalized terminal state constraint. Under mild assumptions, the new strategy is guaranteed to converge in finite time, with arbitrarily good accuracy, to an MPC law with an optimally-chosen terminal state constraint, while still enjoying a larger feasibility set. The features of the new technique are illustrated by an inverted pendulum example in both the tracking and the economic contexts.

Key words: Model predictive control, Economic model predictive control, Constrained control, Optimal control, Nonlinear control

1 Introduction

Model Predictive Control (MPC, see e.g. [24,12]) is one of the few existing techniques that is able to cope, in a quite straightforward way, with the presence of multiple inputs and outputs, of nonlinear dynamics and of hard constraints on the system state, x, and input, u. In MPC, at each time step t the input is computed by solving a finite horizon optimal control problem (FHOCP). The cost function to be minimized in the FHOCP is typically the average, over a finite horizon of $N < \infty$ steps, of the predicted values of a stage cost function, $l(x, u)$. The latter is chosen by the user, according to the goal to be achieved in the control problem at hand. In particular, there are two main classes of problems. In the first class, typically referred to as *tracking MPC*, the aim is to drive the system state and input to reach a given set point or reference trajectory. The stage cost $l(x, u)$ employed in tracking problems is therefore related to the deviation of the predicted state and input trajectories from the reference ones. Most of the existing MPC formulations are concerned with this first class of problems, and a quite vast literature has been developed in the last decades [24], addressing nominal stability and recursive feasibility as well as robustness analysis and robust design (see e.g. [3,13,14,23]). In tracking MPC, the typical way to guarantee recursive feasibility of the FHOCP, as well as asymptotic stability of the target reference trajectory, is the use of a suitable cost function, of a sufficiently long horizon N and/or of "stabilizing constraints", like state contraction constraints [27,5], Lyapunov-like constraints [28], terminal state constraints [20] and terminal set constraints [25].

The second class of problems is that of *economic MPC*, where the stage cost is not directly related to a prescribed set point or trajectory to be tracked, but it expresses a performance to be optimized. Economic MPC is an attractive approach for control problems where the "best" performance, from the point of view of the economic objective, is not attained at any steady state, and/or one wants to avoid the pre-computation of a trajectory to be stabilized with tracking MPC. Economic

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MPC has been applied in practice in various fields, including process control [19,7], renewable energy and energy efficiency [18,4] and transportation [21,29], and the literature concerned with the theoretical properties of economic MPC schemes is all quite recent [6,15,1]. In most of the existing studies, a fixed point (x^s, u^s) is computed that minimizes the average economic cost among all the admissible fixed points. Then, sufficient conditions on the FHOCP are derived, in order to make such a steady state asymptotically stable for the closed-loop system with an economic MPC law. In particular, in [6,1] a terminal state constraint is used, to force the predicted state at time step $t+N$ to be equal to x^s, and conditions on the economic cost function are derived, under which asymptotic stability of (x^s, u^s) is guaranteed. In [1], an asymptotic time-average economic criterion is also introduced, in order to analyze the performance of economic MPC schemes. In [15], the same time-average performance as [1] is considered, but no terminal state constraint is used, and sufficient conditions on the prediction horizon and on the cost function are derived, under which the asymptotic time-average closed-loop performance is "approximately optimal", i.e. it converges to a value close to the minimal one. In [17], a two-mode approach is adopted, by which up to a given time t' an economic performance is optimized while keeping the state trajectory within some level set of a suitable Lyapunov function, then for time $t > t'$ the controller is switched to a more standard tracking mode.
In the described context, we investigate here the use of a terminal state constraint, which we call "generalized" because it requires the state at time step $t+N$ to be equal not to a specific fixed point, e.g. a set point to be tracked or a previously derived optimal fixed point, but to *any* fixed point. Given the same prediction horizon, the use of the generalized terminal state constraint yields in general a larger feasibility set with respect to a classical terminal state equality constraint. We study the links between the use of a standard terminal state equality and a generalized one, in both tracking and economic MPC contexts, and we propose a novel receding horizon algorithm that, under mild assumptions, converges in finite time, with arbitrarily good accuracy, to an MPC law with an optimally chosen terminal state constraint, while still retaining a larger feasibility set. To the best of our knowledge, the idea of using a generalized terminal state constraint has been proposed for the first time in [22,10], in the context of linear systems, and in [9,11], for nonlinear systems, but with different assumptions and with an approach aimed to make the state trajectory always converge to a steady state, which could be not necessarily the best solution in economic MPC problems. The paper is organized as follows. The problem settings are described in section 2; the generalized terminal state constraint, the related FHOCP, its receding horizon implementation and the recursive feasibility property are treated in section 3. Section 4 is concerned with the guaranteed performance of the approach and the novel receding horizon implementation. Finally, we apply the approach to an example involving an inverted pendulum, in both a tracking and an economic problem, in section 5.

2 Notation and problem formulation

We consider discrete-time system models of the form:

$$x(t+1) = f(x(t), u(t)), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $t \in \mathbb{Z}$ is the discrete time variable, $x(t) \in \mathbb{R}^n$ is the system state and $u(t) \in \mathbb{R}^m$ is the input. State constraints are described by a set $\mathbb{X} \subseteq \mathbb{R}^n$, and input constraints by a compact set $\mathbb{U} \subset \mathbb{R}^m$. Mixed state-input constraints can be also considered, but they are omitted here for simplicity. The value of the generic variable y at time $t+j$, predicted at time t, is indicated as $y(j|t)$, $j \in \mathbb{N}$. Let $l : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a stage cost function, let $N \in \mathbb{N}$, $0 < N < \infty$ be a prediction horizon, finally define the cost function J^s as:

$$J^s(x(t), U) \doteq \sum_{j=0}^{N-1} l(x(j|t), u(j|t)), \tag{2}$$

where $U = \{u(0|t), \ldots, u(N-1|t)\}$ is a sequence of N predicted control inputs. Then, the following Finite Horizon Optimal Control Problem (FHOCP) $\mathcal{P}^s(x(t))$ can be formulated:

$$\begin{aligned}
&\mathcal{P}^s(x(t)) : \\
&\min_U J^s(x(t), U) && \text{(3a)}\\
&\text{subject to} \\
&x(j|t) = f(x(j-1|t), u(j-1|t)),\ j = 1, \ldots, N && \text{(3b)}\\
&u(j|t) \in \mathbb{U},\ \forall j = 0, \ldots, N-1 && \text{(3c)}\\
&x(j|t) \in \mathbb{X},\ \forall j = 1, \ldots, N && \text{(3d)}\\
&x(0|t) = x(t) && \text{(3e)}\\
&x(N|t) = x^s, && \text{(3f)}
\end{aligned}$$

where $x^s \in \mathbb{X}$ is fixed and chosen, together with the associated control input $u^s \in \mathbb{U}$, among the (possibly multiple) fixed points (x, u) that minimize the stage cost l (see e.g. [6]):

Definition 1 *(Optimal fixed point)*

$$\begin{array}{c}
(x^s, u^s) \in \arg \min\limits_{x \in \mathbb{X}, u \in \mathbb{U}} l(x, u) \\
\textit{subject to} \\
f(x, u) - x = 0.
\end{array} \tag{4}$$

We note that in (3) we allow values of $x(0)$ outside of the state constraints. If desired, state constraints can be considered also on $x(0)$ by changing eq. (3d) to include also the constraint $x(0|t) \in \mathbb{X}$. All of the following results would remain unchanged in this case.
Problem $\mathcal{P}^s$ is, in general, a nonlinear program (NLP) and, under mild regularity assumptions on f and l, a (possibly local) minimum can be computed

by using a numerical solver, indicated as λ. At a generic time step t, we denote such a solution with $U^*(x(t))$, and the corresponding optimal value with $J^{s*}(x(t)) \doteq J^s(x(t), U^*(x(t)))$.
The feasibility set $\mathcal{F}^s$ is defined as $\mathcal{F}^s \doteq \{x : \mathcal{P}^s(x) \text{ admits a solution}\}$.
Let $\mathcal{B}(r, x) \doteq \{y : \|y - x\|_p \leq r\}$ for some $p \in [1, \infty)$. We consider the following assumption on the set $\mathcal{F}^s$:

Assumption 1 *(Non-emptiness and boundedness of the feasibility set)*
I) $\mathcal{F}^s \neq \varnothing$
II) $\exists r < \infty : \mathcal{F}^s \subset \mathcal{B}(r, 0)$.

Assumption 1 is quite general, since I) holds true if and only if the problem (4) is feasible, i.e. if there exists at least one fixed point that satisfies state and input constraints, and II) is either inherently satisfied by the FHOCP (3), or it can be enforced in most practical applications, where typically the state values that are meaningful for the problem at hand are contained in a bounded set.
In MPC, the FHOCP (3) is solved at each time step by updating the measure of the state variable $x(t)$ according to a receding horizon (RH) strategy:

Algorithm 1 *(RH control with terminal state constraint)*

(1) (initialization) given $x(0) \in \mathcal{F}^s$, *let* $t = 0$, *and solve the FHOCP* $\mathcal{P}^s(x(0))$*; let* $U^*(x(0))$ *be a solution. Apply to the system the control input* $u(0) = u^*(0|0)$. *Set* $t = 1$;
(2) solve the FHOCP $\mathcal{P}^s(x(t))$ *by initializing the solver* λ *with* $\tilde{U} = \{u^*(1|t-1), \ldots, u^*(N-1|t-1), u^s\}$; *let* $U^*(x(t))$ *be a solution;*
(3) apply to the system the control input $u(t) = u^*(0|t)$;
(4) set $t = t + 1$ *and go to (2).*

We denote the state feedback control law, implicitly defined by Algorithm 1, as $u(t) = \kappa^s(x(t))$, $\kappa^s : \mathcal{F}^s \to \mathbb{U}$. In the absence of noise and model uncertainty, for any given initial state $x(0) \in \mathcal{F}^s$, Algorithm 1 guarantees recursive feasibility at all time steps $t > 0$, i.e. $x(t) \in \mathcal{F}^s$, $\forall t > 0$.
The stage cost $l(\cdot, \cdot)$ is chosen according to the considered control problem. In particular, in tracking MPC problems, the function $l(x, u)$ is often chosen as a quadratic function of the state and input tracking errors:

$$l(x, u) = \|x - x^s\|_Q^2 + \|u - u^s\|_R^2, \tag{5}$$

where $\|y\|_M \doteq (y^T M y)^{1/2}$ and $Q = Q^\top$, $R = R^\top$, $Q, R \succ 0$. With this choice (or, more generally, with any function l such that $l(x, u) \geq 0$, $\forall (x, u) \in \mathcal{F}^s \times \mathbb{U}$, and $l(x, u) = 0 \iff (x, u) = (x^s, u^s)$), Algorithm 1 guarantees asymptotic convergence of the state and input to the optimal steady state.
In economic MPC problems, the stage cost l is chosen according to some criterion that has to be minimized (or maximized), e.g. energy loss/production, fuel saving, etc.. In these cases, Algorithm 1 still guarantees recursive feasibility and state and input constraint satisfaction, however convergence and stability properties are not guaranteed in general, since they depend on the properties of the stage cost l. Sufficient conditions for asymptotic stability of the fixed point (x^s, u^s) with an economic stage cost have been derived in [6,1]. However, in economic MPC the stability of (x^s, u^s) may be not relevant with respect to the control objective: in fact, while in tracking MPC the cost to be minimized attains its global minimum at the fixed point (x^s, u^s), which can be regarded as the "best" operating point, in economic MPC the stage cost may not attain its minimum at any steady state, and a steady state solution might not be the most satisfactory operating condition for the system. In [1], an asymptotic time-average economic performance criterion, denoted here as $\overline{J}_\infty$, has been introduced, defined as:

$$\overline{J}_\infty \doteq \lim_{T \to \infty} \sup \frac{\sum_{t=0}^{T} l(x(t), u(t))}{T + 1}. \tag{6}$$

The asymptotic average $\overline{J}_\infty$ appears to be more suited, with respect to stabilization of (x^s, u^s), to represent the control objective in economic MPC problems. Clearly, in closed-loop operation the value of $\overline{J}_\infty$ is a function of the employed control law. In [1], it has been proved that:

$$\overline{J}_\infty(\kappa^s) \leq l(x^s, u^s), \tag{7}$$

thus showing that the use of Algorithm 1 gives an asymptotic time-average economic performance that is better than or equal to that of the stage cost associated to the "best" steady state.
In both tracking and economic MPC, the use of the FHOCP (3) in Algorithm 1 represents a straightforward way to achieve recursive feasibility and constraint satisfaction, however it is well known that the terminal state constraint (3f) can be quite restrictive, so that typically "long" prediction horizons N have to be employed to achieve a satisfactorily large feasibility set $\mathcal{F}^s$, with a consequent higher computational complexity with respect to other techniques, like dual-mode MPC [24]. In this paper, we adopt a particular terminal state constraint, similar to the one proposed e.g. in [22], that aims to reduce this drawback, and we analyze the properties of the resulting closed-loop system in the case of both tracking and economic MPC.

3 Generalized terminal state constraint

Let $V = \{v(0|t), \ldots, v(N|t)\} \in \mathbb{R}^{m \times (N+1)}$ be a sequence of $N + 1$ predicted control inputs, up to time $t + N$, let $\beta \in \mathbb{R}^+$ and $\bar{l}(t) \geq l(x^s, u^s)$ be two scalars, whose role will be better specified later on, and define the cost function J as

$$J(x(t),V) \doteq \sum_{j=0}^{N-1} l(x(j|t),v(j|t)) + \beta l(x(N|t),v(N|t)). \tag{8}$$

Then, we propose to replace the FHOCP (3) with the following:

$$\begin{aligned}
&\mathcal{P}(\bar{l}(t),x(t)) : \\
&\min_V J(x(t),V) && \text{(9a)}\\
&\text{subject to} \\
&x(j|t) = f(x(j-1|t),v(j-1|t)),\ j=1,\ldots,N && \text{(9b)}\\
&v(j|t) \in \mathbb{U},\ \forall j=0,\ldots,N && \text{(9c)}\\
&x(j|t) \in \mathbb{X},\ \forall j=1,\ldots,N && \text{(9d)}\\
&x(0|t) = x(t) && \text{(9e)}\\
&x(N|t) - f(x(N|t),v(N|t)) = 0 && \text{(9f)}\\
&l(x(N|t),v(N|t)) \leq \bar{l}(t). && \text{(9g)}
\end{aligned}$$

We denote a (possibly local) solution of $\mathcal{P}(\bar{l}(t),x(t))$ as $V^*(\bar{l}(t),x(t)) = \{v^*(0|t),\ldots,v^*(N|t)\}$, and the corresponding optimal value as $J^*(\bar{l}(t),x(t)) \doteq J(x(t),V^*(\bar{l}(t),x(t)))$. Moreover, we indicate with $x^*(j|t)$, $j \in [0,N]$ the sequence of predicted state values, computed by using the model (1), starting from $x^*(0|t) = x(t)$ and applying the control sequence $V^*(\bar{l}(t),x(t))$.
The generalized feasibility set $\mathcal{F}$ is defined as $\mathcal{F} \doteq \{x : \mathcal{P}(\bar{l},x)$ admits a solution for some $\bar{l}\}$. For a given $x(t) \in \mathcal{F}$, let us define the set $\mathcal{X}(x(t),N)$ as follows:

Definition 2 *(Set of reachable fixed points)*

$$\begin{aligned}
&\mathcal{X}(x(t),N) \doteq \\
&\{y \in \mathbb{X} : \exists V \in \mathbb{R}^{m\times(N+1)} : v(j|t) \in \mathbb{U},\ \forall j \in [0,N]; \\
&x(N|t) = y;\ f(y,v(N|t)) = y; \\
&x(j|t) = f(x(j-1|t),v(j-1|t)),\ \forall j \in [1,N] \\
&x(j|t) \in \mathbb{X},\ \forall j \in [1,N]\}.
\end{aligned} \tag{10}$$

The set $\mathcal{X}(x(t),N)$ contains all the possible steady state values that can be reached in at most N steps with an admissible control sequence V, starting from $x(t)$. It is straightforward to note that if $N^1 > N^2$, then $\mathcal{X}(x(t),N^1) \supseteq \mathcal{X}(x(t),N^2)$. The following result is concerned with the relationship between the sets $\mathcal{X}(x(t),N)$ and $\mathcal{F}^s$.

Proposition 1 *Let Assumption 1 hold. Then:*

$$x^s \in \mathcal{X}(x(t),N) \iff x(t) \in \mathcal{F}^s \tag{11}$$

PROOF. If $x^s \in \mathcal{X}(x(t),N)$, then by how the set $\mathcal{X}(x(t),N)$ is defined, there exists a sequence V of $N+1$ predicted control moves, such that all of the constraints (9b)-(9f) are satisfied, with $x(N|t) = x^s$. Therefore, the first N elements of such a sequence satisfy also constraints (3a)-(3f), hence $x(t) \in \mathcal{F}^s$. Conversely, if $x(t) \in \mathcal{F}^s$, then there exists an optimal solution $U^*(x(t))$ satisfying constraints (3a)-(3f). Thus, the sequence $V = \{U^*(x(t)),\ u^s\}$ satisfies the constraints in (10) with $x(N|t) = x^s$, i.e. $x^s \in \mathcal{X}(x(t),N)$. □

We also define the quantity $\underline{l}(x(t))$ as:

Definition 3 *(Optimal achievable stage cost) For a given $x(t) \in \mathcal{F}$, the optimal achievable stage cost is:*

$$\begin{aligned}
\underline{l}(x(t)) \doteq &\min_{x\in\mathcal{X}(x(t),N),u\in\mathbb{U}} l(x,u) \\
&\text{subject to} \\
&f(x,u) = x.
\end{aligned} \tag{12}$$

Assumption 2 *(Existence of the optimal achievable cost) For any $x \in \mathcal{F}$, the value $\underline{l}(x(t))$ of Definition 3 exists.*

Assumption 2 holds in most practical cases, considering that the input constraint set $\mathbb{U}$ is compact, the horizon N is finite and the stage cost l can be chosen by the designer.
We can now define the set $\mathcal{S} \doteq \{(\bar{l},x) : \bar{l} \geq \underline{l}(x),\ x \in \mathcal{F}\}$, as well as the following functions:

$$\begin{aligned}
&\kappa(\bar{l}(t),x(t)) \doteq v^*(0|t) \\
&\zeta(\bar{l}(t),x(t)) \doteq l(x^*(N|t),v^*(N|t)) \\
&\kappa : \mathcal{S} \to \mathbb{U} \\
&\zeta : \mathcal{S} \to \mathbb{R}^+
\end{aligned} \tag{13}$$

The value $\kappa(\bar{l}(t),x(t))$ corresponds to the first control input in the sequence $V^*(\bar{l}(t),x(t))$ and the value $\zeta(\bar{l}(t),x(t))$ is the cost associated to the terminal state-input pair, obtained by applying to system (1) the sequence $V^*(\bar{l}(t),x(t))$, starting from the initial condition $x(t)$.
The following RH strategy is obtained by recursively solving the FHOCP (9):

Algorithm 2 *(RH control with generalized terminal state constraint)*

(1) (initialization) choose a value of $\beta > 0$. Given $x(0) \in \mathcal{F}$, choose a value $\bar{l}(0)$ such that $(\bar{l}(0),x(0)) \in \mathcal{S}$ and let $t = 0$. Solve the FHOCP $\mathcal{P}(\bar{l}(0),x(0))$; let $V^*(\bar{l}(0),x(0))$ be a solution. Apply to the system the control input $u(0) = \kappa(\bar{l}(0),x(0))$. Set $t = 1$;
(2) set $\bar{l}(t) = \zeta(\bar{l}(t-1),x(t-1))$ and solve the FHOCP $\mathcal{P}(\bar{l}(t),x(t))$ by initializing the solver λ

with $\tilde{V} = \{v^*(1|t-1), \ldots, v^*(N|t-1), v^*(N|t-1)\}$*; let* $V^*(\bar{l}(t), x(t))$ *be a solution;*

(3) apply to the system the control input $u(t) = \kappa(\bar{l}(t), x(t))$*;*

(4) set $t = t + 1$ *and go to (2).*

Remark 1 *From a practical point of view, we note that the value of* $\underline{l}(x(0))$ *needs not to be known explicitly in the initialization step of Algorithm 2, when selecting* $\bar{l}(0) \in \mathcal{S}$*: in fact, by construction any value of* $\bar{l}(0)$ *such that the problem* $\mathcal{P}(\bar{l}(0), x(0))$ *is feasible belongs to* $\mathcal{S}$*.*

The application of Algorithm 2 gives rise to the following closed-loop system (see Fig. 1):

$$x(t+1) = f(x(t), \kappa(\bar{l}(t), x(t))) \tag{14a}$$

$$\bar{l}(t+1) = \zeta(\bar{l}(t), x(t)) \tag{14b}$$

We denote with $\psi(k, \bar{l}(t), x(t))$ and $\phi(k, \bar{l}(t), x(t))$ the

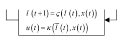

Fig. 1. Closed-loop system obtained with Algorithms 2 and 3.

values of the bound $\bar{l}(t+k)$ and of the state $x(t+k)$, respectively, at the generic time $t+k$, $k \in \mathbb{N}$, obtained by applying (14) starting from $x(t)$ and $\bar{l}(t)$.
Our first result is concerned with the existence of $\mathcal{F}$ and its relationship with the set $\mathcal{F}^s$ and with the properties of recursive feasibility of problem $\mathcal{P}(\bar{l}(t), x(t))$ in Algorithm 2, hence of the capability of the control law κ to satisfy input and state constraints (see also [9] for a similar result under different assumptions and a different proposed approach).

Theorem 1 *Let Assumption 1 hold, and consider the closed-loop system* (14)*, obtained by applying Algorithm 2 with any* $\beta \geq 0$ *in the FHOCP* $\mathcal{P}$*. The following properties hold:*
a) *(feasibility set)*
$\mathcal{F} \supseteq \mathcal{F}^s$.
b) *(recursive feasibility)*
$\mathcal{P}(\psi(t, \bar{l}(0), x(0)), \phi(t, \bar{l}(0), x(0)))$ *is feasible*
$\forall (\bar{l}(0), x(0)) \in \mathcal{S}$, $\forall t > 0$
c) *(state constraint satisfaction)*
$\phi(t, \bar{l}(0), x(0)) \in \mathbb{X}$, $\forall (\bar{l}(0), x(0)) \in \mathcal{S}$, $\forall t > 0$
d) *(input constraint satisfaction)*
$\kappa(\psi(t, \bar{l}(0), x(0)), \phi(t, \bar{l}(0), x(0))) \in \mathbb{U}$, $\forall (\bar{l}(0), x(0)) \in \mathcal{S}$, $\forall t \geq 0$.

PROOF. a) By Assumption 1, there exists a set $\mathcal{F}^s$ of state values such that problem $\mathcal{P}^s$ is feasible. Then, it is straightforward to note that also problem $\mathcal{P}(l(x^s, u^s), x(t))$ is feasible for all $x(t) \in \mathcal{F}^s$.
Points **b)-d)** can be proven with the usual argument of constructing a feasible solution at time t with the tail of the solution computed at time $t-1$, padded with $v^*(N|t-1)$ as the last control input, see e.g. [24]. □

According to Theorem 1, the generalized feasibility set is no smaller than the feasibility set obtained with a fixed terminal state constraint; however nothing can be said, with the assumptions considered so far, about the performance of the closed-loop system obtained by applying Algorithm 2. In fact, the performance and stability guarantees achieved by Algorithm 1, with a fixed terminal state constraint, are a direct consequence of the fact that the employed value of (x^s, u^s) has been optimally chosen off-line, according to the control problem at hand. On the contrary, in Algorithm 2, the terminal state and input $(x^*(N|t), v^*(N|t))$ are different, in general, from the values (x^s, u^s), and they are allowed to change at each time step t. Basically, the values of $(x^*(N|t), v^*(N|t))$ are implicitly "selected", among all the possible steady states that can be reached in at most N steps from the actual state $x(t)$, by the numerical solver λ, in order to minimize the cost $J(x, V)$ (8). Therefore, for given control horizon N and constraints $\mathbb{X}$, $\mathbb{U}$, the values $(x^*(N|t), v^*(N|t))$ depend on the chosen stage cost $l(\cdot, \cdot)$ and on the scalar weight β. Moreover, it can be noted that the use of Algorithm 2 gives rise to a sequence of pairs $\{(x^*(N|t), v^*(N|t))\}_{t=0}^{\infty}$, and consequently a sequence of terminal cost values $\{l(x^*(N|t), v^*(N|t))\}_{t=0}^{\infty}$. The performance achieved by the system (14) clearly depends on the behavior of such a sequence. In this regard, we note that the control input $u(t) = \kappa(\bar{l}(t), x(t))$, obtained by using Algorithm 2, is the output of a dynamical system, with internal state $\bar{l}(t)$ and input $x(t)$ (this is in contrast with the typical MPC control laws, like κ^s, that are static feedback controllers). The controller's state $\bar{l}(t)$ traces the value of the stage cost associated with the terminal state-input pair $l(x^*(N|t), v^*(N|t))$, hence it carries the information about how suboptimal is such a terminal cost with respect to the optimal one, $l(x^s, u^s)$. The inequality $l(x^*(N|t), v^*(N|t)) \leq l(x^*(N|t-1), v^*(N|t-1)) = \bar{l}(t)$, enforced by means of constraint (9g), ensures that the sequence $\{l(x^*(N|t), v^*(N|t))\}_{t=0}^{\infty}$ is not increasing, however, in the general settings considered so far, there is no guarantee of convergence to the optimal value $l(x^s, u^s)$, or to a value close to the optimal. One option to deal with this issue is to use Algorithm 2 as it is, to set some initial choices of N, l and β and to tune these parameters following a trial-and-error procedure, in order to obtain a satisfactory closed-loop performance. Indeed, quite good results can be typically obtained in this way. Another option is to consider additional assumptions on the problem, in order to derive guidelines on how to choose N, l and β, as well as to adopt a more sophisticated receding horizon algorithm, in order to guarantee a desired behavior of the sequence $\{l(x^*(N|t), v^*(N|t))\}_{t=0}^{\infty}$, in terms of convergence to a value which is arbitrarily close to the optimal one, $l(x^s, u^s)$. Such a modified algorithm and its properties are described in the next section.

4 Guaranteed properties of MPC with generalized terminal state constraint

We first establish sufficient conditions on β under which the terminal state and input pair $(x^*(N|t), v^*(N|t))$, computed by solving problem $\mathcal{P}(\bar{l}(t), x(t))$, has an associated cost $l(x^*(N|t), v^*(N|t))$ which is arbitrarily close to the minimal one, among all the possible steady states that can be reached from $x(t)$. In order to do so, we consider the next three assumptions. We recall that a continuous, monotonically increasing function $\alpha : [0, +\infty) \to [0, +\infty)$ is a $\mathcal{K}_\infty$ function if $\alpha(0) = 0$ and $\lim_{a\to+\infty} \alpha(a) = +\infty$, and denote such functions as $\alpha \in \mathcal{K}_\infty$.

Assumption 3 *(Boundedness of the generalized feasibility set and continuity of f and l)*
I) $\exists r < \infty : \mathcal{F} \subset \mathcal{B}(r, 0)$;
II) f and l are continuous on $\overline{\mathcal{F}} \times \mathbb{U}$, where $\overline{\mathcal{F}}$ is the closure of $\mathcal{F}$, hence $\exists\, \alpha_f, \alpha_l \in \mathcal{K}_\infty : \|f(\bar{x}, \bar{u}) - f(\hat{x}, \hat{u})\| \leq \alpha_f(\|(\bar{x}, \bar{u}) - (\hat{x}, \hat{u})\|)$, $|l(\bar{x}, \bar{u}) - l(\hat{x}, \hat{u})| \leq \alpha_l(\|(\bar{x}, \bar{u}) - (\hat{x}, \hat{u})\|)$, $\forall (\bar{x}, \bar{u}), (\hat{x}, \hat{u}) \in \mathcal{F} \times \mathbb{U}$, for some vector norm $\|\cdot\|$.

Assumption 4 *(Solution of the FHOCP $\mathcal{P}$)*
For any $(\bar{l}, x) \in \mathcal{S}$, the FHOCP $\mathcal{P}(\bar{l}, x)$ has at least one global minimum, which is computed by the solver λ independently on how it is initialized.

Assumption 4 is quite usual and implicitly considered in the context of economic MPC and nonlinear MPC. Moreover, it is satisfied if problem $\mathcal{P}(\bar{l}, x)$ is convex, which is the important case of MPC for linear systems with convex constraints $\mathbb{X}$, $\mathbb{U}$ and convex stage cost l.

Assumption 5 *(Stage cost)*
There exists a set $\mathcal{M} \subset \mathcal{F} \times \mathbb{U}$ where the function l attains its minimum. Without loss of generality, $l(x, u) \geq 0$, $\forall (x, u) \in \mathcal{F} \times \mathbb{U}$, and $l(x, u) = 0 \iff (x, u) \in \mathcal{M}$.

Assumption 5 essentially states that the stage cost function l (which is continuous in virtue of Assumption 3) must have a global minimum inside the cartesian product of the feasibility set (which is bounded) with the input constraints set (which is compact). Such a minimum can be attained at some point or set of points. This Assumption is obviously satisfied for tracking MPC with stage costs like (5), with $\mathcal{M} = \{(x^s, u^s)\}$. In the case of economic MPC, satisfaction of Assumption 5 depends on the stage cost chosen by the control designer, and the set $\mathcal{M}$ often does not contain any steady state and might also be not connected. We show an example of the latter case in section 5.
We can now derive a result related to the optimality of the pair $(x^*(N|t), v^*(N|t))$ with respect to the stage cost function l.

Proposition 2 *Let Assumptions 1-4 hold. Then, for any $\epsilon > 0$, there exists a finite value $\underline{\beta}(\epsilon)$ such that, for any given $x(t) \in \mathcal{F}$ and any $\bar{l}(t) \geq \underline{l}(x(t)) + \epsilon$, if $\beta \geq \underline{\beta}(\epsilon)$ then*

$$l(x^*(N|t), v^*(N|t)) \leq \underline{l}(x(t)) + \epsilon \tag{15}$$

where $(x^*(N|t), v^*(N|t))$ are the optimal terminal state and input computed by solving problem $\mathcal{P}(\bar{l}(t), x(t))$.

PROOF. Let $(\overline{x}, \overline{u})$ be a state-input pair such that:

$$(\overline{x}, \overline{u}) = \arg \min_{x \in \mathcal{X}(x(t), N), u \in \mathbb{U}} l(x, u) \quad \text{subject to} \quad f(x, u) = x,$$

and let $\overline{V}$ be a sequence of $N + 1$ control inputs which satisfies constraints (9b)-(9f) and such that $x(N|t) = \overline{x}$, $v(N|t) = \overline{u}$. This sequence is guaranteed to exist by Definition 2. Moreover, we have $l(\overline{x}, \overline{u}) = \underline{l}(x(t)) < \bar{l}(t)$, hence also constraint (9g) is satisfied and the sequence $\overline{V}$ is admissible for problem $\mathcal{P}(\bar{l}(t), x(t))$. The cost associated with $\overline{V}$ is equal to:

$$J(x(t), \overline{V}) = \sum_{j=0}^{N-1} l(\overline{x}(j|t), \overline{v}(j|t)) + \beta \underline{l}(x(t)),$$

where $\overline{x}(j|t)$, $j \in [0, N]$ is the state trajectory obtained by applying the sequence $\overline{V}$. Consider now any other possible state $\hat{x} \in \mathcal{X}(x(t), N)$ and input $\hat{u} \in \mathbb{U}$ such that $f(\hat{x}, \hat{u}) = \hat{x}$, $l(\hat{x}, \hat{u}) > \underline{l}(x(t)) + \epsilon$ and $l(\hat{x}, \hat{u}) \leq \bar{l}(t)$. Denote with $\hat{V}$ a sequence of control inputs which is feasible for problem $\mathcal{P}(\bar{l}(t), x(t))$ and such that $x(N|t) = \hat{x}$. Then, the cost associated with $\hat{V}$ is:

$$J(x(t), \hat{V}) = \sum_{j=0}^{N-1} l(\hat{x}(j|t), \hat{v}(j|t)) + \beta l(\hat{x}, \hat{u}),$$

where $\hat{x}(j|t)$, $j \in [0, N]$ is the state trajectory obtained by applying the sequence $\hat{V}$. Thus, the difference $J(x(t), \overline{V}) - J(x(t), \hat{V})$ is given by:

$$\begin{aligned} &J(x(t), \overline{V}) - J(x(t), \hat{V}) = \beta[\underline{l}(x(t)) - l(\hat{x}, \hat{u})] + \\ &\sum_{j=0}^{N-1} [l(\overline{x}(j|t), \overline{v}(j|t)) - l(\hat{x}(j|t), \hat{v}(j|t))], \end{aligned} \tag{16}$$

and, by exploiting Assumption 3, it holds:

$$J(x(t), \overline{V}) - J(x(t), \hat{V}) < -\beta\epsilon + \eta, \tag{17}$$

with (see the Appendix for a complete proof)

$$\eta = \sum_{i=0}^{N-1} \sum_{j=0}^{i} \alpha_l \left(\alpha_f^{(i-j)} \left(\max_{\overline{v}, \hat{v} \in \mathbb{U}} \|\overline{v} - \hat{v}\| \right) \right) > 0, \quad (18)$$

where $\alpha_f^{(i)}(a) \doteq \underbrace{\alpha_f(\alpha_f(\ldots \alpha_f(a) \ldots))}_{i \text{ times}}$ and $\alpha^{(0)}(a) \doteq a$. Thus, by setting $\underline{\beta}(\epsilon) = \eta/\epsilon$ and $\beta \geq \underline{\beta}(\epsilon)$, the inequality

$$J(x(t), \hat{V}) > J(x(t), \overline{V}) \quad (19)$$

is obtained. Note that, due to the compactness of $\mathbb{U}$, the value of η is finite, hence also $\underline{\beta}(\epsilon)$ is finite. Now assume, with the purpose of contradiction, that the solution V^* to the FHOCP is such that $l(x^*(N|t), v^*(N|t)) > \underline{l}(x(t)) + \epsilon$. Then, inequality (19) would hold true with $\hat{V} = V^*(\bar{l}(t), x(t))$, meaning that the cost associated to $V^*(\bar{l}(t), x(t))$ would be higher than the one associated to $\overline{V}$. However, this cannot happen, since, by Assumption 4, $V^*(\bar{l}(t), x(t))$ is such that $J(x(t), V^*(\bar{l}(t), x(t))) = J^*(\bar{l}(t), x(t)) \leq J(x(t), V)$ for any V which is feasible for problem $\mathcal{P}(\bar{l}(t), x(t))$. Hence, the inequality (15) must hold. □

Proposition 2 induces a result pertaining to the performance achieved by using Algorithm 2 when the initial state $x(0)$ belongs to the feasibility set $\mathcal{F}^s$. Before stating such a result, we consider the following assumption for tracking MPC schemes.

Assumption 6 *(Stage cost in tracking MPC)*
In tracking MPC, the stage cost l enjoys the following properties:
I) (global minimum)

$$\begin{aligned} &l(x, u) > 0, \, \forall (x, u) \in \mathcal{F}^s \times \mathbb{U} \setminus \{(x^s, u^s)\} \\ &l(x^s, u^s) = 0 \end{aligned} \quad (20)$$

II) (lower bound)

$$\begin{aligned} &\exists \underline{\alpha}_l \in \mathcal{K}_\infty : \\ &\underline{\alpha}_l(\|(x, u) - (x^s, u^s)\|) \leq l(x, u), \, \forall (x, u) \in \mathcal{F}^s \times \mathbb{U}. \end{aligned} \quad (21)$$

Note that Assumption 6 is typically satisfied by the stage cost functions used in tracking MPC, like (5).

Theorem 2 *Let Assumptions 1-5 hold, let a value of $\epsilon > 0$ be chosen, and let $\beta \geq \underline{\beta}(\epsilon)$. For any $x(0) \in \mathcal{F}^s$, apply Algorithm 2. Then, the following properties hold:*
a) *(sub-optimality of the terminal stage cost)*

$$l(x^*(N|t), v^*(N|t)) - l(x^s, u^s) \leq \epsilon, \, \forall t \geq 0, \quad (22)$$

b) *(tracking MPC) if Assumption 6 also holds, then:*

$$\|(x^*(N|t), v^*(N|t)) - (x^s, u^s)\| \leq \underline{\alpha}_l^{-1}(\epsilon), \, \forall t \geq 0, \quad (23)$$

c) *(economic MPC) the asymptotic average performance obtained by control law κ is bounded as:*

$$\overline{J}_\infty(\kappa) \leq l(x^s, u^s) + \epsilon. \quad (24)$$

PROOF. a) According to Proposition 1, if $x(0) \in \mathcal{F}^s$ then $x^s \in \mathcal{X}(x(0), N)$. Moreover, by Definitions 1 and 3, if $x^s \in \mathcal{X}(x(0), N)$ then $\underline{l}(x(0)) = l(x^s, u^s)$. Therefore, by Proposition 2 we have $l(x^*(N|0), v^*(N|0)) - l(x^s, u^s) \leq \epsilon$. The use of Algorithm 2 and constraint (9g) force the values $l(x^*(N|t), v^*(N|t))$ to be not increasing with t, thus the inequality $l(x^*(N|t), v^*(N|t)) - l(x^s, u^s) \leq \epsilon$ holds true for all $t \geq 0$.
b) From (22), under Assumption 6-I) we have
$l(x^*(N|t), v^*(N|t)) - l(x^s, u^s)$
$= l(x^*(N|t), v^*(N|t)) \leq \epsilon, \, \forall t \geq 0.$
Then, by Assumption 6-II) it holds $\|(x^*(N|t), v^*(N|t)) - (x^s, u^s)\| \leq \underline{\alpha}_l^{-1}(\epsilon), \, \forall t \geq 0.$
c) The proof of this claim follows that of Theorem 1 in [1], with little modifications, and it is reported here for the sake of completeness. First of all, note that, for any $t \geq 0$, it holds:

$$J^*(\bar{l}(t), x(t)) = l(x(t), u(t)) + \sum_{j=1}^{N-1} l(x^*(j|t), v^*(j|t)) + \beta l(x^*(N|t), v^*(N|t));$$

$$J^*(\bar{l}(t+1), x(t+1)) \leq \sum_{j=1}^{N-1} l(x^*(j|t), v^*(j|t)) + l(x^*(N|t), v^*(N|t)) + \beta l(x^*(N|t), v^*(N|t)),$$

thus, under Assumptions 1-5, by using (22), for any $x(0) \in \mathcal{F}^s$ it holds:

$$\begin{aligned} &J^*(\bar{l}(t+1), x(t+1)) - J^*(\bar{l}(t), x(t)) \\ &\leq l(x^s, u^s) + \epsilon - l(x(t), u(t)) \\ \Rightarrow &\lim_{T \to \infty} \inf \frac{\sum_{t=0}^{T} J^*(\bar{l}(t+1), x(t+1)) - J^*(\bar{l}(t), x(t))}{T+1} \\ \leq &\lim_{T \to \infty} \inf \frac{\sum_{t=0}^{T} l(x^s, u^s) + \epsilon - l(x(t), u(t))}{T+1} \\ = &\, l(x^s, u^s) + \epsilon - \lim_{T \to \infty} \sup \frac{\sum_{t=0}^{T} l(x(t), u(t))}{T+1}. \end{aligned}$$

At the same time, due to Assumption 5,

$$\begin{aligned} &\lim_{T \to \infty} \inf \frac{\sum_{t=0}^{T} J^*(\bar{l}(t+1), x(t+1)) - J^*(\bar{l}(t), x(t))}{T+1} \\ = &\lim_{T \to \infty} \inf \frac{J^*(\bar{l}(T+1), x(T+1)) - J^*(\bar{l}(0), x(0))}{T+1} \\ \geq &\lim_{T \to \infty} \inf \frac{-J^*(\bar{l}(0), x(0))}{T+1} = 0 \end{aligned}$$

hence $\lim_{T\to\infty}\sup\frac{\sum_{t=0}^{T} l(x(t),u(t))}{T+1} = \overline{J}_\infty(\kappa) \leq l(x^s,u^s)+\epsilon$. This bound establishes the result. □

According to Theorem 2-**a)**, with a sufficiently large value of β, for any $x(0) \in \mathcal{F}^s$, the generalized terminal stage cost is always at most ϵ-suboptimal with respect to the one related to the optimal pair (x^s,u^s) (4). This result provides a clear indication on how to tune the scalar β, which is the only additional design parameter with respect to a standard approach with fixed terminal state constraint: as β is increased, the cost associated with the terminal state/input pair gets closer to the best one. On the other hand, an overly large value of β might give rise to numerical problems in the solution of the FHOCP and cause poor performance during transients, since the cost function tends to be dominated by the terminal stage cost. Hence, a possible tuning guideline could be to start with some value of β, e.g. equal to 1 (i.e. giving to the terminal stage the same weight as the other predicted stages), and then to gradually increase it, until a satisfactory closed-loop behavior is obtained. Note that the approach will guarantee constraint satisfaction and recursive feasibility with any value of β. More insights on the role of β, as well as a strategy for self-tuning this parameter, have been recently proposed in [26].
Theorem 2-**b)** implies that it is possible to force the generalized terminal state-input pair to be arbitrarily close, as $\epsilon \to 0$, to the desired one, for any $x(0) \in \mathcal{F}^s$. As a consequence, the convergence and stability properties of tracking MPC schemes, with a fixed terminal state constraint, can be extended to the case of generalized terminal state constraint, by considering an arbitrarily small neighborhood of the desired set point (x^s,u^s).
Finally, Theorem 2-**c)** states that the MPC scheme with generalized terminal state constraint achieves an asymptotic average performance which is better than that of the optimal fixed point (x^s,u^s), plus an arbitrarily small tolerance ϵ.
Practically speaking, Theorem 2 states that the control law $\kappa(x)$ can achieve closed-loop properties that are arbitrarily close to those of $\kappa^s(x)$, for any $x(0) \in \mathcal{F}^s$. For state values $x(0) \in \mathcal{F} \setminus \mathcal{F}^s$, such a comparison can not be made, since the control law κ^s is not defined outside the set $\mathcal{F}^s$, and the optimal fixed point x^s is not reachable (see Proposition 1). We now focus our attention on initial state values $x(0) \in \mathcal{F}\setminus\mathcal{F}^s$, and we present a modified algorithm that guarantees that the resulting MPC law enjoys the properties of Theorem 2, under the following additional assumption.

Assumption 7 *(Sequences of steady states with decreasing stage cost)*
For some (eventually very small) $\overline{\epsilon} > 0$ there exists a minimal prediction horizon $\underline{N} \in \mathbb{N}$ such that, for any fixed point $(\overline{x},\overline{u}) \in \mathcal{F}\times\mathbb{U} : f(\overline{x},\overline{u}) = \overline{x}$, there exists at least one state-input pair $(\tilde{x},\tilde{u})$ with the following properties:
I) $f(\tilde{x},\tilde{u}) = \tilde{x}$;
II) $\tilde{x} \in \mathcal{X}(\overline{x},\underline{N})$;
III) $l(\tilde{x},\tilde{u}) \leq \max\left(l(x^s,u^s), l(\overline{x},\overline{u})-\overline{\epsilon}\right)$.

The practical meaning of Assumption 7 is the following: for any fixed point $(\overline{x},\overline{u})$ in the set $\mathcal{F}\times\mathbb{U}$, there exists another fixed point $(\tilde{x},\tilde{u})$ for the dynamics (1) (property I)), belonging to the set of reachable fixed points $\mathcal{X}(\overline{x},\underline{N})$ (property II)). The value of the stage cost $l(\tilde{x},\tilde{u})$ is either equal to the minimal one among all fixed points, $l(x^s,u^s)$, or it is strictly lower, at least by $\overline{\epsilon}$, than $l(\overline{x},\overline{u})$ (property III)). We note that, if Assumption 1 holds, Assumption 7 is clearly satisfied at least for any fixed point $(\overline{x},\overline{u}) \in \mathcal{F}^s\times\mathbb{U}$, with $\underline{N} = N$ and $(\tilde{x},\tilde{u}) = (x^s,u^s)$.
The modified MPC algorithm with generalized terminal state constraint is given below.

Algorithm 3 *(Modified RH control with generalized terminal state constraint)*

(1) (initialization) Select an arbitrarily small value of $\epsilon > 0$ such that $2\epsilon \leq \overline{\epsilon}$, and select $\beta \geq \underline{\beta}(\epsilon)$. Given $x(0) \in \mathcal{F}$, choose a value $\bar{l}(0)$ such that $(\bar{l}(0),x(0)) \in \mathcal{S}$ and let $t = 0$. Solve the FHOCP $\mathcal{P}(\bar{l}(0),x(0))$; let $V^*(\bar{l}(0),x(0))$ be a solution. Apply to the system the control input $u(0) = v^*(0|0)$. Set $t = 1$;
(2) set $\bar{l}(t) = \zeta(\bar{l}(t-1),x(t-1))$, solve the FHOCP $\mathcal{P}(\bar{l}(t),x(t))$ by initializing the solver λ with $\tilde{V} = \{v^*(1|t-1),\dots,v^*(N|t-1),v^*(N|t-1)\}$; let $V^*(\bar{l}(t),x(t))$ be a solution;
(3) if $l(x^*(N|t),v^*(N|t)) > \bar{l}(t)-\epsilon$ and $l(x^*(N|t),v^*(N|t)) > l(x^s,u^s)+\epsilon$, then set $V^*(\bar{l}(t),x(t)) = \tilde{V}$ and, consequently, $(x^*(N|t),v^*(N|t)) = (x^*(N|t-1),v^*(N|t-1))$;
(4) apply the control input $u(t) = \kappa(\bar{l}(t),x(t))$;
(5) set $t = t+1$ and go to (2).

The next Theorem shows that the use of Algorithm 3 produces a sequence of terminal stage costs $\{l(x^*(N|t),v^*(N|t))\}_{t=0}^{\infty}$ that converges in finite time, within the arbitrarily small tolerance ϵ, to the optimal value $l(x^s,u^s)$.

Theorem 3 *Let Assumptions 1-5 and 7 hold, and consider the closed-loop system obtained by applying Algorithm 3 with $N \geq \underline{N}$. Then, for any value of $x(0) \in \mathcal{F}$ there exists a finite number of time steps $\overline{T}(x(0))$ such that:*

$$l(x^*(N|\overline{T}(x(0))),v^*(N|\overline{T}(x(0)))) \leq l(x^s,u^s)+\epsilon \quad (25)$$

PROOF. Consider any $x(0) \in \mathcal{F}$. If $x(0) \in \mathcal{F}^s$, then by Theorem 2-**a)** we have $l(x^*(N|0),v^*(N|0)) \leq l(x^s,u^s)+\epsilon$, $\forall t \geq 0$, hence (25) also holds with $\overline{T}(x(0)) = 0$. If $x(0) \in \mathcal{F}\setminus\mathcal{F}^s$, then an optimal terminal state-input pair, $(x^*(N|0),v^*(N|0))$, is computed, and

the control input $u(0) = \kappa(\bar{l}(0), x(0)) = v^*(0|0)$ is applied to the system. Consider now a generic time step $t > 0$: assuming that $l(x^*(N|t), v^*(N|t)) > l(x^s, u^s) + \epsilon$, the following two cases may occur.
(a) if $l(x^*(N|t), v^*(N|t)) \leq \bar{l}(x(t)) - \epsilon$, then, considering that $\bar{l}(x(t)) = \zeta(\bar{l}(t-1), x(t-1)) = l(x^*(N|t-1), v^*(N|t-1))$, the optimal terminal stage cost decreases at least by the quantity ϵ:
$l(x^*(N|t), v^*(N|t)) - l(x^*(N|t-1), v^*(N|t-1)) \leq -\epsilon$.
(b) if $l(x^*(N|t), v^*(N|t)) > \bar{l}(x(t)) - \epsilon$, then at step (3) of Algorithm 3 the solution $V^*(\bar{l}(t), x(t))$ of $\mathcal{P}(\bar{l}(t), x(t))$ is replaced by the tail of the previously computed optimal solution. Let us denote with τ the last time step at which the solution $V^*(\bar{l}(\tau), x(\tau))$ of $\mathcal{P}(\bar{l}(\tau), x(\tau))$ was retained. Therefore, the control input at time t is $u(t) = v^*(t-\tau|\tau)$, and the trajectory of the system evolves according to the optimal one predicted at time step τ. The same procedure is carried out as long as case (b) holds true, until the state $x(\tau+N) = x^*(N|\tau)$ is eventually reached, which happens at most in N time steps. In virtue of Assumption 7, since $N \geq \underline{N}$, once the fixed point $(x(\tau+N), u(\tau+N)) = (x^*(N|\tau), v^*(N|\tau))$ has been reached, there exists at least one fixed point $(\tilde{x}, \tilde{u})$, such that $\tilde{x} \in \mathcal{X}(x(\tau+N), N)$ and $l(\tilde{x}, \tilde{u}) \leq \max\left(l(x^s, u^s), l(x^*(N|\tau), v^*(N|\tau)) - \bar{\epsilon}\right)$. Hence, by Definition 3 we have $\underline{l}(x(\tau+N)) \leq l(\tilde{x}, \tilde{u}) \leq \max\left(l(x^s, u^s), l(x^*(N|\tau), v^*(N|\tau)) - \bar{\epsilon}\right)$. Now, if $\max\left(l(x^s, u^s), l(x^*(N|\tau), v^*(N|\tau)) - \bar{\epsilon}\right) = l(x^s, u^s)$, then $\underline{l}(x(\tau+N)) = l(x^s, u^s)$ and, by Proposition 2, we have $l(x^*(N|\tau+N), v^*(N|\tau+N)) \leq l(x^s, u^s) + \epsilon$, thus the result (25) holds true. If, on the contrary, $\max\left(l(x^s, u^s), l(x^*(N|\tau), v^*(N|\tau)) - \bar{\epsilon}\right) = l(x^*(N|\tau), v^*(N|\tau)) - \bar{\epsilon}$, then
$\underline{l}(x(\tau+N)) \leq l(x^*(N|\tau), v^*(N|\tau)) - \bar{\epsilon}$ and, again by Proposition 2, we have
$l(x^*(N|\tau+N), v^*(N|\tau+N)) \leq \underline{l}(x(\tau+N)) + \epsilon \leq l(x^*(N|\tau), v^*(N|\tau)) - \bar{\epsilon} + \epsilon$. Since $2\epsilon \leq \bar{\epsilon}$, the inequality $l(x^*(N|\tau+N), v^*(N|\tau+N)) \leq l(x^*(N|\tau), v^*(N|\tau)) - \epsilon$ holds true, i.e. after at most N time steps, the terminal stage cost decreases:
$l(x^*(N|\tau+N), v^*(N|\tau+N)) - l(x^*(N|\tau), v^*(N|\tau)) \leq -\epsilon$.
Summing up, while either cases (a) or (b) occur, i.e. as long as condition $l(x^*(N|t), v^*(N|t)) > l(x^s, u^s) + \epsilon$ holds true, the quantity $l(x^*(N|t+N), v^*(N|t+N))$ generally decreases (not strictly) with t. In particular, the decrease is guaranteed to be always at least equal to $-\epsilon$, and to take place at most every N time steps:

$$\begin{aligned}
&\forall j \geq 1, \\
&l(x^*(N|jN), v^*(N|jN)) \\
-\ &l(x^*(N|(j-1)N), v^*(N|(j-1)N)) \leq -\epsilon.
\end{aligned} \tag{26}$$

Moreover, due to Assumption 3 we have:

$$\begin{aligned}
&l(x^*(N|0), v^*(N|0)) - l(x^s, u^s) \\
\leq\ &\alpha_l(\|(x^*(N|0), v^*(N|0)) - (x^s, u^s)\|).
\end{aligned} \tag{27}$$

Equations (26)-(27) lead to the following result:

$$\begin{aligned}
&l(x^*(N|jN), v^*(N|jN)) - l(x^s, u^s) \\
\leq\ &\alpha_l(\|(x^*(N|0), v^*(N|0)) - (x^s, u^s)\|) - j\epsilon,
\end{aligned}$$

hence when $j > \frac{\alpha_l(\|(x^*(N|0), v^*(N|0)) - (x^s, u^s)\|)}{\epsilon}$ the condition $l(x^*(N|jN), v^*(N|jN)) \leq l(x^s, u^s) + \epsilon$ is guaranteed to be satisfied. Therefore, we have

$$l(x^*(N|\overline{T}(x(0))), v^*(N|\overline{T}(x(0)))) \leq l(x^s, u^s) + \epsilon,$$

where

$$\overline{T}(x(0)) = N \left\lceil \frac{\alpha_l(\|(x^*(N|0), v^*(N|0)) - (x^s, u^s)\|)}{\epsilon} \right\rceil,$$

and $\lceil \cdot \rceil$ denotes the ceiling operation to the closest integer. Note that the pair $(x^*(N|0), v^*(N|0))$ is a function of the initial state $x(0)$ only, and thus also the quantity $\overline{T}(x(0))$ is. □

Remark 2 *According to Theorem 3, under the considered Assumptions, for any initial state $x(0)$ inside the feasibility set $\mathcal{F}$, by using Algorithm 3 the stage cost of the terminal state-input pair converges to a value that is arbitrarily close to the optimal one, $l(x^s, u^s)$, after at most a finite number $\overline{T}(x(0))$ of time steps. Then, it can be noted that all the properties of Theorem 2 hold true also if Algorithm 3 is used, for all time steps $t \geq \overline{T}(x(0))$.*

Remark 3 *At step (3) of Algorithm 3, we use the tail of the previously computed optimal control sequence just for the sake of simplicity. One other option could be, at any time step t such that the condition $l(x^*(N|t), v^*(N|t)) > \bar{l}(t) - \epsilon$ and $l(x^*(N|t), v^*(N|t)) > l(x^s, u^s) + \epsilon$ is detected, to use an auxiliary MPC scheme, designed to reach the terminal state $(x^*(N|t-1), v^*(N|t-1))$ in finite time. In this case, Theorem 3 would still hold with minor modifications.*

Theorem 3 provides a guideline similar to the one obtained by Proposition 2 for tuning β: with a sufficiently large value (and with a long enough prediction horizon as required by Assumption 7) convergence of the cost associated with the terminal state/iput pair to a value close to optimal is achieved. This finding is consistent with the results of related work concerned with MPC without terminal state constraints and with a weighting factor on the terminal stage cost, see e.g. [16]. As we anticipated in the introduction, the idea of letting the terminal steady state be a decision variable has been proposed already in [22,10] for linear systems and in [9,11] for nonlinear ones. However, the results of [9,11] require more restrictive assumptions on f and l, namely Lipschitz continuity, and do not account for cases in which a large enough value of N has to be used to guarantee global results, as we consider with our Assumption 7. This is important e.g. if the set containing the steady-states is not connected (see section 5.1 for an example).

Finally, the approach proposed in [11] for economic MPC renders the best steady state x^s asymptotically stable, hence the resulting asymptotic average economic performance is generally worse than the one obtained with the technique proposed here, which is able to obtain non-steady-state trajectories in favor of better economic performance (we provide an example of this aspect in section 5.2).

5 Numerical examples

We consider an inverted pendulum whose equations of motion written in normalized variables are (see [2]):

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \sin(x_1(t)) - u(t)\cos(x_1(t)). \end{aligned} \tag{28}$$

The input constraint set $\mathbb{U}$ is $\mathbb{U} = \{u \in \mathbb{R} : |u| < 0.5\}$.

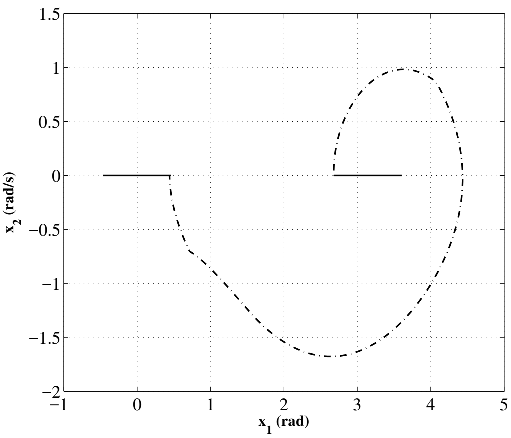

Fig. 2. Inverted pendulum example with tracking MPC: set of admissible steady states (solid lines) and state trajectory achievable in $N = 141$ steps from the initial steady state $x = [\pi - \arctan(.5),\ 0]^T$.

The state variables are the pendulum angular position x_1 (modulo 2π) and angular speed x_2. The following discrete time model to be used in the MPC design is obtained by forward difference approximation of (28):

$$\begin{aligned} x_{1,t+1} &= x_{1,t} + T_s x_{2,t} \\ x_{2,t+1} &= x_{2,t} + T_s\ (\sin(x_{1,t}) - u_t \cos(x_{1,t})) \end{aligned} \tag{29}$$

with sampling time $T_s = 0.05\,\text{s}$. We choose the following stage cost:

$$l(x,u) = [\sin(x_1/2)\,,\ x_2]\,Q \begin{bmatrix} \sin(x_1/2) \\ x_2 \end{bmatrix} + u^2\,R, \tag{30}$$

with $Q = \begin{bmatrix} 225 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 1$. We present here two examples that share the same model (29), input constraints and cost function: the first one pertains to a tracking problem, the second to an economic one. More examples are given in [8].

5.1 Tracking MPC

In this first example, we want to track the open-loop unstable fixed point $[x_1, x_2, u]^T = [0, 0, 0]^T$, starting from the downright position $[x_1, x_2, u]^T = [\pi, 0, 0]^T$. The set of all admissible fixed points for the system is given by:

$$\{(\overline{x}, \overline{u}) : \sin(\overline{x}_1) - \overline{u}\cos(\overline{x}_1) = 0,\ \overline{x}_2 = 0,\ |\overline{u}| < 0.5\}\,.$$

It can be noted that such a set is not connected, in particular its projection on the state space (depicted in Fig. 2, solid lines) is given by:

$$\begin{aligned} &\{x : x_1 \in [-\arctan(0.5),\ \arctan(0.5)] \cup \\ &[\pi - \arctan(0.5),\ \pi + \arctan(0.5)];\ x_2 = 0\}\,. \end{aligned}$$

In this situation, the use of a generalized terminal state constraint might not be able to drive the system state to the target $x^s = 0$, starting from a steady state $\overline{x}$ such that $\overline{x}_1 \in [\pi - \arctan(0.5), \pi + \arctan(0.5)]$, unless a sufficiently large horizon N is chosen, as stated in our Assumption 7. A value of N that satisfies this Assumption is $N = 141$, as shown in Fig. 2 (dash-dot line) by the related predicted trajectory starting from the initial steady state $\overline{x} = [\pi - \arctan(.5),\ 0]^T$ and input $\overline{u} = -0.5$. However, note that since Assumption 7 is only sufficient for Theorem 3 to hold, also lower values of N might give the desired results. In particular, we show here the results with $N = 100$ and $\beta = 100$. The controller is able to swing up the pendulum in about 12.5 s. The state trajectory in the phase plane is depicted in Fig. 3 (solid black line), together with the trajectories predicted at each time step and the corresponding terminal steady states. In particular, it can be noted how the sequence of terminal state-input pairs, $(x^*(N|t), u^*(N|t))$, is equal to $(\pi + \arctan(.5), 0)$ for $t < 3\,\text{s}$, then it jumps to $(\pi - \arctan(.5), 0)$ for $t \in [3\,\text{s},\ 6.6\,\text{s}]$ and finally converges to the target steady state, after about 132 time steps (i.e. 6.6 s). These numerical values depend on the initial state, on the input constraints and on the chosen stage cost function and prediction horizon.
We also applied a tracking MPC law with a fixed terminal state constraint, i.e. $x(N|t) = 0$. With the considered input constraint, the corresponding FHOCP (3) results to be unfeasible for horizons $N < 200$, i.e. twice the one used with the generalized terminal state constraint. The state trajectory obtained with the fixed terminal state constraint is shown in Fig. 3 (dash-dot black line). This controller is able to swing up the pendulum in about 11 s. Therefore, this example confirms that 1. given the same prediction horizon, the use of a

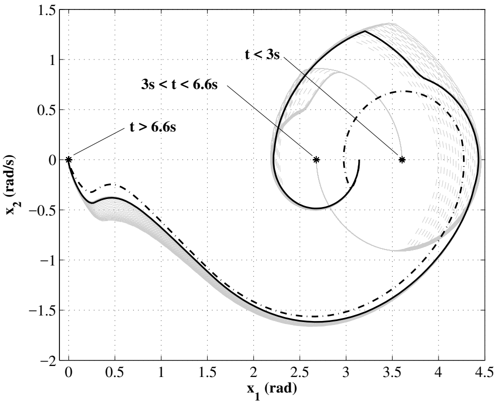

Fig. 3. Inverted pendulum example with tracking MPC: closed loop state trajectory obtained with the generalized terminal state constraint (solid black line), with $N = 100$, and trajectories predicted at each time step (gray dashed lines). Predicted terminal states are marked with '$*$'. Dash–dot black line: closed loop state trajectory obtained with a fixed terminal state constraint, and $N = 200$.

generalized terminal state constraint can give a feasibility set which is larger than that obtained with a fixed, optimally-chosen terminal state constraint, and 2. that the performance obtained with the generalized terminal state constraint, in terms of swing-up time, are very close to those achieved with a fixed terminal state constraint, but the required prediction horizon (i.e. computational effort) is much shorter.

5.2 *Economic MPC*

In this second example, we consider $N = 60$ and we include the following state constraints:

$$X = \left\{ x \in \mathbb{R}^2 : \begin{array}{c} \frac{\pi}{3} \leq x_1 \leq \frac{5\pi}{3} \\ -2 \leq x_2 \leq 2 \end{array} \right\}. \quad (31)$$

In this case, the steady state/input $[x_1, x_2, u]^T = [0, 0, 0]^T$ is outside the constraints (31). The solutions to problem (4), i.e. the best feasible steady states, are the values $[x_1^s, x_2^s, u^s]^T = [\pi + \arctan(0.5), 0, 0.5]^T$ and $[x_1^s, x_2^s, u^s]^T = [\pi - \arctan(0.5), 0, -0.5]^T$, which have the same economic cost of 213.37. We select $x^s = [\pi - \arctan(0.5), 0]^T$ and $u^s = -0.5$. However, this steady-state does not minimize the stage cost for all state/input pairs inside the feasibility set: as an example, the point $[x_1^s, x_2^s, u^s]^T = [2.5, 0, 0]^T$, which is not a steady-state but lies within the feasibility set, has a better stage cost, equal to 202.62. This feature renders the problem an economic one, since the best stage cost is not attained at any feasible steady state. We apply our approach to this example, choosing $\beta = 20$. The obtained results are reported in Figure 4: the state trajectory of the closed loop system converges to a periodic orbit, whose asymptotic average cost is equal to 198.25, i.e. better than that of (x^s, u^s). We

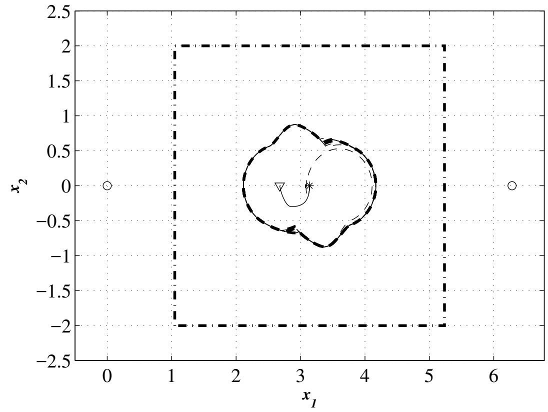

Fig. 4. Inverted pendulum example with economic MPC. Closed-loop state trajectories obtained with the approach described in this paper (dashed line) and with the technique of [11] (solid line). The steady state x^s to which the system controlled with the approach of [11] converges is indicated by '∇'. Thick dashed line: periodic trajectory to which the system controlled by the approach described in this paper converges. Thick dash-dot line: state constraints. The initial condition $[x_1, x_2]^T = [\pi, 0]$ is indicated by '$*$', while the (unfeasible) state value where the economic cost achieves its minimum is indicated by '$\circ$'.

note that the terminal state/input pair does not reach a fixed value in this case, rather it jumps periodically between the two fixed points $[\pi + \arctan(0.5), 0, 0.5]^T$ and $[\pi - \arctan(0.5), 0, -0.5]^T$, which have the same (optimal) associated stage cost. To provide a comparison, we apply also the approach of [11] to this example. Since the stage cost (30) by itself does not satisfy the strong duality property required in [11], following the approach of [6] we add a quadratic penalty to it:

$$l_P(x, u) = [\sin(x_1/2),\ x_2]\, Q \begin{bmatrix} \sin(x_1/2) \\ x_2 \end{bmatrix} + u^2 R + P\,(\|x_1 - x_1^s\|_2^2 + |u - u^s|^2), \quad (32)$$

with $P = 55$. Problem (4) with the modified stage cost (32) enjoys strong duality; the resulting rotated stage cost is obtained with the multiplier $\lambda = [0\ -900]^T$. Finally, we choose the same control horizon $N = 60$ and the offset function (see [11] for details) $V_O = 10^3 \|(x(N_c), u(N_c)) - (x^s, u^s)\|_2^2$. The obtained results are shown in Figure 4: as expected, the state converges to the best achievable steady state x^s. The related asymptotic average economic cost is the one corresponding to such a steady state, i.e. 213.37. Therefore, this example shows that our approach is different from the technique proposed in [11] and that it can give rise to solutions with better asymptotic average economic performance with respect to the best achievable steady state.

6 Conclusions

We investigated a generalized terminal state constraint for Model Predictive Control and proved that, under reasonable assumptions, the resulting closed-loop system has performance similar to those of MPC schemes with a fixed, optimally chosen terminal state constraint, while enjoying a larger feasibility set given the same prediction horizon. The approach introduces only one additional scalar tuning parameter with respect to a standard MPC technique with fixed terminal state constraint. With respect to previous works in the literature, which introduced the use of such a generalized terminal state constraint, the technique proposed here requires less demanding assumptions, it is able to deal also with non-connected sets of steady states and it yields in general better asymptotic average performance in economic MPC problems. These features have been highlighted through an inverted pendulum example.

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Appendix

Proof of equations (17)-(18). Consider equation (16). The term $\beta[\underline{l}(x(t))-l(\hat{x},\hat{u})]$ is less than $-\beta\epsilon$ because $\beta > 0$ and the pair $(\hat{x},\hat{u})$ is such that $l(\hat{x},\hat{u}) > \underline{l}(x(t))+\epsilon$. The terms $[l(\overline{x}(i|t),\overline{v}(i|t)) - l(\hat{x}(i|t),\hat{v}(i|t))]$, $i \in [0, N-1]$ can be bounded on the basis of Assumption 3, as follows:

case: $i = 0$;

$$\begin{aligned}
& l(\overline{x}(0|t),\overline{v}(0|t)) - l(\hat{x}(0|t),\hat{v}(0|t)) \\
\leq\ & |l(\overline{x}(0|t),\overline{v}(0|t)) - l(\hat{x}(0|t),\hat{v}(0|t))| \\
\leq\ & \alpha_l(\|\overline{v}(0|t) - \hat{v}(0|t)\|) \\
=\ & \sum_{j=0}^{i} \alpha_l \left(\alpha_f^{(i-j)}(\|\overline{v}(j|t) - \hat{v}(j|t)\|)\right);
\end{aligned}$$

case: $i = 1$;

$$\begin{aligned}
& l(\overline{x}(1|t),\overline{v}(1|t)) - l(\hat{x}(1|t),\hat{v}(1|t)) \\
=\ & l(\overline{x}(1|t),\overline{v}(1|t)) - l(\hat{x}(1|t),\overline{v}(1|t)) \\
& +l(\hat{x}(1|t),\overline{v}(1|t)) - l(\hat{x}(1|t),\hat{v}(1|t)) \\
\leq\ & |l(\overline{x}(1|t),\overline{v}(1|t)) - l(\hat{x}(1|t),\overline{v}(1|t))| \\
& +|l(\hat{x}(1|t),\overline{v}(1|t)) - l(\hat{x}(1|t),\hat{v}(1|t))| \\
\leq\ & \alpha_l(\|\overline{x}(1|t) - \hat{x}(1|t)\|) + \alpha_l(\|\overline{v}(1|t) - \hat{v}(1|t)\|) \\
\leq\ & \alpha_l(\alpha_f(\|\overline{v}(0|t) - \hat{v}(0|t)\|)) + \alpha_l(\|\overline{v}(1|t) - \hat{v}(1|t)\|) \\
=\ & \sum_{j=0}^{i} \alpha_l \left(\alpha_f^{(i-j)}(\|\overline{v}(j|t) - \hat{v}(j|t)\|)\right);
\end{aligned}$$

...

generic i

$$\begin{aligned}
& l(\overline{x}(i|t),\overline{v}(i|t)) - l(\hat{x}(i|t),\hat{v}(i|t)) \\
\leq\ & \sum_{j=0}^{i} \alpha_l \left(\alpha_f^{(i-j)}(\|\overline{v}(j|t) - \hat{v}(j|t)\|)\right).
\end{aligned}$$

Thus, the second term in (16) can be bounded by

$$\eta = \sum_{i=0}^{N-1} \sum_{j=0}^{i} \alpha_l \left(\alpha_f^{(i-j)}\left(\max_{\overline{v},\hat{v}\in\mathbb{U}} \|\overline{v} - \hat{v}\|\right)\right). \qquad \square$$

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