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Learning-based predictive control for MIMO systems

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***Abstract*— A learning-based approach for robust predictive control design for multi-input multi-output (MIMO) linear systems is presented. The identification stage allows to obtain multi-step ahead prediction models and to derive tight uncertainty bounds. The identified models are then used by a robust model predictive controller, that is designed for the tracking problem with stabilizing properties. The proposed algorithm is used to control the nonlinear model of a quadruple-tank process using data gathered from it. The resulting controller, suitably modified to account for the nonlinear system gain matrix, results in remarkable tracking performances.**

I. INTRODUCTION

Given the growing complexity of systems and plants, and the easy availability of large datasets collected during plant operation, researchers are focusing more and more on learning techniques to extract knowledge from data [5]. This trend involves also the control community, that has devoted attention to learning and identification algorithms and tools for dynamical system modeling [15], [19]. Data-driven modeling methods allow to overcome most of the critical issues of the approaches based on first principle equations, e.g. lack of information on the system or the complexity of the phenomena. Furthermore, data-driven approaches often allow to save valuable time in industrial applications.

Several techniques have been developed so far with the aim of designing a controller from data. They are either *model-free*, i.e. they derive the control policy directly from data [20], [7], [18] or *model-based*, i.e. they first derive a model of the plant, based on which the control design is addressed [3],[2],[10]. A critical issue of model-based approaches is the quantification of the uncertainty associated with the learned model. Having a quantitative description of the uncertainty is crucial to quantify the mismatch between the model and the real system, that must be taken into account by the controller. The most common choice, though, is to assume the uncertainty bound known [4], [11], [8], but few approaches are devoted to derive it [12]. In recent years, a number of contributions has addressed this problem exploiting set membership techniques [13], that are promising in this context since they allow to quantify the model uncertainty from data [14].

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Among the most widely employed control design methods model predictive control (MPC) [17] has gained popularity due to the large flexibility and the capability of handling operational constraints. It naturally requires a model of the system, based on which an optimization program is formulated, according to the desired objective, and periodically solved, to compute a proper control action [9]. Recently, a novel approach has been proposed for linear systems, that addresses model identification, model mismatch quantification, and MPC design in a unitary and consistent fashion [22]. First of all, set membership identification [21] is used to derive multi-step prediction models and for quantifying their associated uncertainty bounds. Specifically tailored robust MPC controllers are also proposed for regulation [23] and tracking [22], which are able to include explicitly multi-step predictors and prediction error bounds into the MPC controller formulation. These contributions focus on single input and single output (i.e., SISO) systems, and they have been tested on linear academic examples taken from the related literature.

In this paper we proceed along this line of research. In particular, we propose an extension of the approach described in [22] for multi input multi output (MIMO) systems and we test the controller on a realistic benchmark example, i.e., the quad-tank system [1]. Motivated by this example, we also discuss about some issues and possible solutions arising in the application of our approach on nonlinear systems.

In Section II the problem is formally stated, in Section III the control algorithm is devised, while a short description of the learning algorithm is given in Section IV. In Section V the numerical results are reported. Conclusions and hints for future work conclude the paper in Section VI.

Notation. We denote with I_n the identity matrix of order n, with $0_{a,b}$ the matrix of dimensions $a \times b$ with all entries equal to 0. Also, we denote with $\mathbf{1}_n$ the vector with n entries equal to 1 and with $\otimes$ the Kronecker product. The Minkowski sum and the Pontryagin difference between sets are denoted with $\oplus$ and with $\ominus$ respectively. Given a matrix A, we denote $A_{i\bullet}$ its i-th row and $A_{\bullet i}$ its i-th column. The 2-norm of a vector v is denoted with $\|v\| = \sqrt{v^T v}$, its i-th component is denoted with v_i and its infinity norm is $\|v\|_\infty = \max_i |v_i|$. Finally, given two numbers $m, n \in \mathbb{N}$, we define with $rem^*(m/n)$ the least positive remainder of the division m/n: however, in case m is multiple of n, we set $rem^*(m/n) = n$ rather than 0.

II. PROBLEM FORMULATION

Consider a discrete-time, linear time invariant, MIMO system of order n with input $u \in \mathbb{R}^{n_u}$, output $z \in \mathbb{R}^{n_y}$ and measured output $y \in \mathbb{R}^{n_y}$ described by the autoregressive equations:

$$\begin{array}{rcl} z(k+1) & = & \theta_z^{(1)^T} \phi_z^{(1)}(k) + v(k) \\ y(k) & = & z(k) + d(k), \end{array} \quad (1)$$

where $v \in \mathbb{R}^{n_y}$ is an additive process disturbance, $d \in \mathbb{R}^{n_y}$ is an additive measurement noise and $\phi_z^{(1)}(k) \in \mathbb{R}^{(n_y+n_u)n}$ is the regressor defined as:

$$\begin{array}{rcl} \phi_z^{(1)}(k) & = & [z^T(k), \dots, z^T(k-n+1), u^T(k-1), \dots, \\ & & u^T(k-n+1), u^T(k)]^T \end{array}$$

The matrix $\theta_z^{(1)}$ contains the real system parameters, that are unknown, as well as the order of the system n.

Assumption 1 (System and signals):

- System (1) is asymptotically stable
- $u(k) \in \mathbb{U} \subset \mathbb{R}^{n_u} \, \forall k \in \mathbb{Z}$, where $\mathbb{U}$ is a compact and convex set.
- the process disturbances $v(k)$ and the output noises $d(k)$ are bounded, i.e.

  $$|v_i(k)| \leq \bar{v}_i \quad |d_i(k)| \leq \bar{d}_i \quad \forall k \in \mathbb{Z}, \forall i = 1, \dots, n_y$$

  where $\bar{d}_i > 0$ are known and $\bar{v}_i > 0$ are possibly not known (and assumed unknown in the sequel).

In this paper, for simplicity, we also assume that the system is square (i.e., $n_u = n_y$) and that the input-output static gain matrix (denoted μ in the paper) is invertible.

A dataset collected from the plant is available, composed of N_s input-output pairs (u, y), and it is assumed that the input signal in the database excites all the system modes to ensure its identifiability.

The problem addressed in this paper consists of designing a theoretically sound robust model predictive controller based on identified multi-step prediction models, endowed with a tight prediction error bound estimated from data.

III. TRACKING MPC WITH LEARNED MODELS

In this section we extend the procedure for the design of an MPC robust controller for tracking proposed in [22] to MIMO systems.

A. Available models

The proposed approach requires the definition of a state-space simulation model with state $X(k) = [z^T(k), \dots, z^T(k-o+1), u^T(k-1), \dots, u^T(k-o+1)]^T \in \mathbb{R}^{(n_y+n_u)o-n_u}$ and dynamics

$$\begin{aligned} X(k+1) &= AX(k) + Bu(k) + Mw(k) \\ z(k) &= CX(k) \\ y(k) &= z(k) + d(k) \end{aligned} \quad (2)$$

where o (see later) is selected in the identification phase and the entries $w_i(k)$ of $w(k) \in \mathbb{R}^{n_y}$ are bounded, i.e., $|w_i(k)| \leq \bar{w}_i$. Note that $w(k)$ accounts for various sources of uncertainty, e.g., the model mismatch with the real system and the process disturbances. Also, our approach requires the definition of optimal predictors of $z(k+p)$ (denoted $z_p(k)$), for $p = 1, \dots, \bar{p}$, based on the present system state and on the future control actions $U(k) = \begin{bmatrix} u(k)^T & \dots & u(k+\bar{p})^T \end{bmatrix}^T$:

$$z_p(k) = C_p X(k) + D_p U(k) \quad (3)$$

Model (2) will be defined through a suitable learning phase. Also, the peculiarity of our approach lies in the fact that (3) will not be defined by iterating (2) p times, but by means of dedicated identification steps, in order to optimize their multi-step predictive potentialities.

For consistency of notation we define $C_0 = C$ and $D_0 = 0_{n_y,(\bar{p}+1)n_u}$ such that we can write $z(k) = z_0(k) = C_0 X(k) + D_0 U(k)$.

A dedicated observer is designed, that includes the estimate $\hat{w}(k)$ of the disturbance w, derived later on.

$$\hat{X}(k+1) = A\hat{X}(k) + Bu(k) + M\hat{w}(k) + L(y(k) - C\hat{X}(k)) \quad (4)$$

$\hat{X}(k)$ is the estimated state and the matrix L is chosen such that the closed-loop matrix $(A - LC)$ is Schur stable. We also define the nominal dynamic system as

$$\bar{X}(k+1) = A\bar{X}(k) + B\bar{u}(k) + M\hat{w}(k) \quad (5)$$

where

$$u(k) = \bar{u}(k) + K(\hat{X}(k) - \bar{X}(k)) \quad (6)$$

The gain K is defined in such a way that the closed-loop transition matrix $A + BK$ is Schur stable. The corresponding nominal outputs are, for all $p = 1, \dots, \bar{p}$

$$\bar{z}_p(k) = C_p \bar{X}(k) + D_p \bar{U}(k) \quad (7)$$

where $\bar{U}(k) = \begin{bmatrix} \bar{u}^T(k) & \dots & \bar{u}^T(k+\bar{p}) \end{bmatrix}^T$. We finally define $\bar{z}(k) = C_0 \bar{X}(k) = \bar{z}_0(k)$. In line with [11], the optimization problem will be formulated by regarding the nominal model (5), while the displacement of the real variables $X(k)$, $u(k)$ with respect to $\bar{X}(k)$, $\bar{u}(k)$ will be considered for constraint tightening purposes.

B. Definition of the cost function

The overall goal of the proposed method is to track the reference output z_{goal}. However, for feasibility purposes, in the optimization problem the reference set point is z_{ref}, which in turn is defined as a further optimization variable. Assuming that a reliable (invertible) estimate $\hat{\mu}$ of the system static gain matrix μ is available, the corresponding input and state references are

$$u_{\text{ref}}(k) = \hat{\mu}^{-1} z_{\text{ref}}(k), \quad X_{\text{ref}}(k) = N z_{\text{ref}}(k) \quad (8)$$

where $N = \begin{bmatrix} \mathbf{1}_o \otimes I_{n_y} \\ \mathbf{1}_{o-1} \otimes \hat{\mu}^{-1} \end{bmatrix}$. As a result, the estimate $\hat{w}(k)$ of $w(k)$ is defined according to the steady-state expression $X_{\text{ref}}(k) = AX_{\text{ref}}(k) + Bu_{\text{ref}}(k) + M\hat{w}(k)$, i.e.,

$$\hat{w}(k) = \eta_{zw} z_{\text{ref}}(k) \quad (9)$$

where $\eta_{zw} = M^T \left[(I_{(n_y+n_u)o-n_u} - A)N - B\hat{\mu}^{-1} \right]$. Moreover, for consistency, the term $\hat{w}(k)$ will be forced to be bounded, i.e., $|\hat{w}_i(k)| \leq \bar{w}_i$ for all $i = 1, \dots, n_y$ through dedicated constraints in the optimization problem.

From this we can define, $\forall p \in \{0, \dots, \bar{p}\}$, the consistent set point for each p-steps ahead prediction model as

$$z^p_{\text{ref}}(k) = \begin{bmatrix} C_p & D_p \end{bmatrix} \begin{bmatrix} X_{\text{ref}}(k) \\ \mathbf{1}_{\bar{p}+1} \otimes u_{\text{ref}}(k) \end{bmatrix} \tag{10}$$

The cost function to be minimized at each step k is

$$J(k) = \sum_{p=0}^{\bar{p}} \left(\|\bar{z}_p(k) - z^p_{\text{ref}}(k)\|^2_{Q_p} + \|\bar{u}(k+p) - u_{\text{ref}}(k)\|^2_{R_p} \right) + \|\bar{X}(k+\bar{p}+1) - X_{\text{ref}}(k)\|^2_P + \sigma \|z_{\text{ref}}(k) - z_{\text{goal}}\|^2 \tag{11}$$

where $\bar{X}(k+\bar{p}+1)$ is obtained by iterating the unperturbed state equation (5) $\bar{p}+1$ times, i.e.,

$$\bar{X}(k+\bar{p}+1) = A^{\bar{p}+1}\bar{X}(k) + \Gamma \bar{U}(k) + \Gamma_w(\mathbf{1}_{\bar{p}+1} \otimes \hat{w}(k)) \tag{12}$$

Also, $\Gamma = \begin{bmatrix} A^{\bar{p}}B & \dots & B \end{bmatrix}, \Gamma_w = \begin{bmatrix} A^{\bar{p}}M & \dots & M \end{bmatrix}$, The weights Q_p, R_p, P, and $\sigma > 0$ are defined to guarantee convergence properties, see [22] for details.

C. Definition of the tightened constraints

Suitable tightened input and output constraints are imposed on variables $\bar{u}(k)$ and $\bar{z}(k) = C\bar{X}(k)$

$$\bar{u}(k) \in \bar{\mathbb{U}}, \quad \bar{z}(k) \in \bar{\mathbb{Z}}, \quad \hat{w}(k) \in \mathbb{W}, \tag{13}$$

where $\mathbb{W} = \{w \in \mathbb{R}^{n_y} : |w_i| \leq \bar{w}_i, i = 1, \dots, n_y\}$ and the sets $\bar{\mathbb{U}}$ and $\bar{\mathbb{Z}}$ are closed and satisfy:

$$\bar{\mathbb{U}} \subseteq \mathbb{U} \ominus K\bar{\mathbb{E}} \tag{14a}$$

$$\bar{\mathbb{Z}} \subseteq \mathbb{Z} \ominus C(\bar{\mathbb{E}} \oplus \hat{\mathbb{E}}) \tag{14b}$$

Set $\hat{\mathbb{E}}$ is robust positively invariant (RPI) [16] for the system

$$\hat{e}(k+1) = (A - LC)\hat{e}(k) + M(w(k) - \hat{w}(k)) - Ld(k) \tag{15}$$

Note that (15) describes the evolution of $\hat{e}(k) = X(k) - \hat{X}(k)$. Also, $\bar{\mathbb{E}}$ is the RPI set for

$$\bar{e}(k+1) = (A + BK)\bar{e}(k) + LC\hat{e}(k) + Ld(k) \tag{16}$$

where $\bar{e}(k) = \hat{X}(k) - \bar{X}(k)$. To define the terminal constraint set we consider the following auxiliary control law

$$\bar{u}(k) = u_{\text{ref}}(k) + K(\bar{X}(k) - X_{\text{ref}}(k)) \tag{17}$$

To compute an invariant set where $(\bar{X}(k), z_{\text{ref}})$ must lie in order to guarantee that constraints (13) are verified for all k, we need to define the Maximal Output Admissible Set (MOAS, see [6]) $\mathbb{O}$ for the system

$$\begin{bmatrix} \bar{X}(k+1) \\ z_{\text{ref}}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A + BK & BM_2 + M\eta_{zw} \\ 0_{n_y,(n_u+n_y)o-n_u} & I_{n_y} \end{bmatrix}}_{\mathcal{F}} \begin{bmatrix} \bar{X}(k) \\ z_{\text{ref}}(k) \end{bmatrix} \tag{18}$$

that is subject to the auxiliary control law (17), where $M_2 = \hat{\mu}^{-1} - KN$. The triplet $(\bar{u}(k), \bar{z}(k), \hat{w}(k))$ is computed as

$$\begin{bmatrix} \bar{z}(k) \\ \bar{u}(k) \\ \hat{w}(k) \end{bmatrix} = \underbrace{\begin{bmatrix} C & 0_{n_y,n_y} \\ K & M_2 \\ 0_{n_y,(n_u+n_y)o-n_u} & \eta_{zw} \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} \bar{X}(k) \\ z_{\text{ref}}(k) \end{bmatrix} \tag{19}$$

In the following we will use the invariant, polytopic inner approximation $\mathbb{O}_\epsilon$ to the MOAS.

D. The optimization problem

The optimization problem, to be solved at each time instant $k \geq 0$, reads

$$\min_{\bar{X}(k),\bar{U}(k),z_{\text{ref}}(k)} J(k) \tag{20a}$$

subject to (5), (7), (8), (9), (10) and

$$\hat{X}(k) - \bar{X}(k) \in \bar{\mathbb{E}} \tag{20b}$$

Also, $\forall p \in \{0, \dots, \bar{p}\}$

$$\bar{u}(k+p) \in \bar{\mathbb{U}}, \quad \bar{z}(k+p) = C_0\bar{X}(k+p) \in \bar{\mathbb{Z}}, \quad \hat{w}(k) \in \mathbb{W} \tag{20c}$$

Finally, as a terminal constraint, the following must be fulfilled

$$\begin{bmatrix} \bar{X}(k+\bar{p}+1) \\ z_{\text{ref}}(k) \end{bmatrix} \in \mathbb{O}_\epsilon \tag{20d}$$

If available, the solution to the optimization problem (20) is denoted $\bar{X}(k|k), \bar{U}(k|k) = (\bar{u}(k|k), \dots, \bar{u}(k+\bar{p}|k)), z_{\text{ref}}(k|k)$, and $u(k)$ in (6) is applied to the system according to the receding horizon principle. Theorem 1 in [22] guarantees convergence and recursive feasibility.

IV. LEARNING MULTI-STEP MODELS

A. Definition of multistep prediction models in normal form

In this section we discuss how to obtain multistep models and uncertainty bounds by adapting the original algorithm in [21], [22] in case of MIMO systems.

In particular, we first note that, by iteration of (1), the p-steps ahead output value of the real system can be computed. In particular, defining the sequence $\mathcal{V}(k) = [v^T(k), \dots, v^T(k+\bar{p}-1)]^T$ and the extended regressor vector, containing also future inputs up to time $k+p-1$, $\phi_z^{(p)}(k) =$

$$=[z^T(k), \dots, z^T(k-n+1), u^T(k-1), \dots, u^T(k-n+1), u^T(k), u^T(k+1), \dots, u^T(k+p-1)]^T$$

it is possible write:

$$z(k+p) = {\theta_z^{(p)}}^T \phi_z^{(p)}(k) + {\theta_v^{(p)}}^T \mathcal{V}(k) \tag{21}$$

In (21) the matrices $\theta_z^{(p)}$ and $\theta_v^{(p)}$ are polynomial combinations of the entries of $\theta_z^{(1)}$, and they are thus unknown. By introducing the vector $\mathcal{D}(k) = [d^T(k), d^T(k+1), \dots, d^T(k+\bar{p}-1)]^T$, expression (21) can be re-written as a function of the regressor $\phi_y^{(p)}(k)$, which corresponds to $\phi_z^{(p)}(k)$ where z samples have been replaced by y ones. The resulting expression reads:

$$z(k+p) = {\theta_y^{(p)}}^T \phi_y^{(p)}(k) + \underbrace{{\theta_d^{(p)}}^T \mathcal{D}(k) + {\theta_v^{(p)}}^T \mathcal{V}(k)}_{\mathcal{H}(k)} \tag{22}$$

Inspired by the form of (22), that provides directly the p-steps ahead value of output vector z from $\phi_y^{(p)}(k)$, we select an independent model to predict the output vector at each value of $p = 1, \dots, \bar{p}$. Each one of these predictors, that we term "multistep", is of the following form:

$$\hat{z}(k+p) \quad = \quad \hat{\theta}^{(p)^T} \tilde{\phi}_y^{(p)}(k) \tag{23}$$

In (23) $\hat{\theta}^{(p)^T}$ is a matrix $\in \mathbb{R}^{n_y \times (n_y+n_u)o+n_u(p-1)}$, where o is the chosen model order, so that $\tilde{\phi}_y^{(p)}(k) = [y^T(k), \dots, y^T(k-o+1), u^T(k-1), \dots, u^T(k-o+1), u^T(k), u^T(k+1), \dots, u^T(k+p-1)]^T$. Note that, given the definition of $\tilde{\phi}_y^{(p)}(k)$, the matrix $\hat{\theta}^{(p)^T}$ can be naturally partitioned into submatrices referred to variable y (denoted with $\hat{\theta}_Y^{(p)}$), to past inputs (denoted with $\hat{\theta}_{\bar{U}}^{(p)}$) and to current and future inputs (denoted with $\hat{\theta}_U^{(p)}$), so that $\hat{z}(k+p) = \begin{bmatrix} \hat{\theta}_Y^{(p)^T} & \hat{\theta}_{\bar{U}}^{(p)^T} & \hat{\theta}_U^{(p)^T} \end{bmatrix} \tilde{\phi}_y^{(p)}(k)$. Model (23), though, is not in a form that is suitable for the direct application of the identification algorithm in [22], which requires the model parameters to be in a vector.

We thus reformulate (23) in a more convenient way. In particular, considering that $\hat{\theta}_{\bullet i}^{(p)}, i = 1, \dots, n_y$ contains the set of parameters associated to the i-th output, we write

$$\hat{z}(k+p) = \underbrace{\begin{bmatrix} \tilde{\phi}_y^{(p)^T}(k) & 0 & \dots & 0 \\ 0 & \tilde{\phi}_y^{(p)^T}(k) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & \tilde{\phi}_y^{(p)^T}(k) \end{bmatrix}}_{\tilde{\Phi}_y^{(p)T}} \underbrace{\begin{bmatrix} \hat{\theta}_{\bullet 1}^{(p)} \\ \hat{\theta}_{\bullet 2}^{(p)} \\ \vdots \\ \hat{\theta}_{\bullet n_y}^{(p)} \end{bmatrix}}_{\hat{\Theta}^{(p)}} \tag{24}$$

In equation (24) the predictor contains a unique vector of unknown parameters to be identified, and it thus fits the form required by the algorithm in [22], to which the reader is referred for details.

The outcome of the identification procedure consists, for each prediction step $p = 1, \dots, \bar{p}$ in:

- A nominal model vector, denoted with $\hat{\Theta}^{*(p)}$. It can be straightforwardly recast as in (23), and we denote this representation with:

$$\hat{\theta}^{*(p)^T} = \begin{bmatrix} \hat{\theta}_Y^{*(p)^T} & \hat{\theta}_{\bar{U}}^{*(p)^T} & \hat{\theta}_U^{*(p)^T} \end{bmatrix}$$

.

- A global prediction error bound for each one of the system outputs, i.e. a vector $\hat{\tau}^{(p)}(\cdot) \in \mathbb{R}^{n_y}$ such that:

$$\begin{aligned} &|z_j(k+p) - \hat{z}_j(k+p)| \leq \hat{\tau}_j^{(p)}(\hat{\Theta}^{*(p)}), \\ &\forall k \in \mathbb{Z},\, j = 1, \dots, n_y \end{aligned}$$

These bounds are termed global since they depend only on the vector of parameters (i.e. the model) and not on the specific regressor. They are valid over a compact set of regressor trajectories of interest, and they enjoy asymptotic convergence properties under suitable assumptions on the informative content of the data collected in such a compact set, see [22] for details.

B. Definition of the control-oriented models

The models (2) and (3) are obtained by setting

$$A = \begin{bmatrix} \hat{\theta}_Y^{*(1)^T} & \hat{\theta}_{\bar{U}}^{*(1)^T} \\ I_{(o-1)n_y} \quad 0_{(o-1)n_y, n_y} & 0_{(o-1)n_y,(o-1)n_u} \\ 0_{(o-1)n_u, on_y} & \begin{matrix} 0_{n_u,(o-1)n_u} \\ I_{(o-2)n_u} \quad 0_{(o-2)n_u,n_u} \end{matrix} \end{bmatrix},$$

$$B = \begin{bmatrix} \hat{\theta}_U^{*(1)^T} \\ 0_{(o-1)n_y,n_u} \\ I_{n_u} \\ 0_{(o-2)n_u,n_u} \end{bmatrix}, C = \begin{bmatrix} I_{n_y} & 0_{n_y,(n_y+n_u)(o-1)} \end{bmatrix}, M = C^T$$

$$C_p = \begin{bmatrix} \hat{\theta}_Y^{*(p)^T} & \hat{\theta}_U^{*(p)^T} \end{bmatrix},\ D_p = \begin{bmatrix} \hat{\theta}_{\bar{U}}^{*(p)^T} & 0_{n_y,(\bar{p}+1-p)n_u} \end{bmatrix}.$$

The maximum amplitude of each of the components $\bar{w}_i$ is identified based on the prediction error bound $\hat{\tau}_j^{(p)}(\cdot)$ suitably employed, with arguments similar to those presented in [22], with special attention to the multivariable nature of the system. In particular, we obtain $\bar{w} = [\bar{w}_1, \dots, \bar{w}_{n_y}]^T$ as a solution to optimization program (25) stated below. Specifically, $\forall p = 1, \dots, \bar{p}$, $\hat{\Theta}^{*(1,p)}$ represents the 1 step nominal model iterated p times according to (2), $c \in \mathbb{R}^{1,n_y}$ is a weighting vector, and $E = \begin{bmatrix} I_{on_y} \\ 0_{(o-1)n_u,on_y} \end{bmatrix}$ is a selection matrix.

$$\begin{aligned} &\bar{w} = \arg \min_{w_i \in \mathbb{R}^+} c^T w \qquad\qquad (25) \\ &\text{s.t.} \quad \sum_{i=0}^{p-1} \sum_{h=1}^{n_y} \left| C_{j\bullet} A^i M_{\bullet h} \right| w_h + \\ &\quad + \sum_{i=1}^{on_y} |C_{j\bullet} A^p E_{\bullet i}| \bar{d}_{rem^*(i/n_y)} \quad \geq \hat{\tau}_j^{(p)}(\hat{\Theta}^{*(1,p)}) \\ &\quad \forall j = 1, \dots, n_y,\ \forall p = 1, \dots, \bar{p} \end{aligned}$$

V. NUMERICAL EXAMPLE

In this section we test the proposed approach on a quadruple-tank system. After a short discussion on the issues arising in the application of the proposed approach to a nonlinear system, we show the results obtained with a nonlinear simulator.

A. The quad-tanks case study

We consider the system proposed in [1], consisting of a four-tank system. The water level of the tanks are denoted $h_1, \dots, h_4$ and their dynamics is described by the following model.

$$\begin{aligned} S\dot{h}_1(t) &= -a_1 \sqrt{2gh_1(t)} + a_3 \sqrt{2gh_3(t)} + \gamma_a q_a(t), \\ S\dot{h}_2(t) &= -a_2 \sqrt{2gh_2(t)} + a_4 \sqrt{2gh_4} + \gamma_b q_b(t), \\ S\dot{h}_3(t) &= -a_3 \sqrt{2gh_3(t)} + (1-\gamma_b) q_b(t), \\ S\dot{h}_3(t) &= -a_4 \sqrt{2gh_4(t)} + (1-\gamma_a) q_a(t), \end{aligned} \tag{26}$$

where a_i (with $i = 1, \dots, 4$) is the discharge constant of the i-th tank, S is the cross section of the tanks and g is the gravitational acceleration. The inputs to the system are flowrates q_a and q_b, generated by two pumps. Tanks 1 and 4 are filled with flowrates $\gamma_a q_a$ and $(1-\gamma_a) q_a$, respectively, where $\gamma_a \in [0, 1]$ is the aperture ratio of the three-way valve

TABLE I

PARAMETERS OF THE PLANT.

	a_1	a_2	$h_{\max}$	$h_{\min}$	γ_a	γ_b
value	$1.31 \cdot 10^{-4}$	$1.51 \cdot 10^{-4}$	1.3	0.3	0.3	0.4
unit	m^2	m^2	m	m	-	-
	a_3	a_4	$q_{\max}$	$q_{\min}$	S	
value	$9.57 \cdot 10^{-4}$	$8.82 \cdot 10^{-4}$	3	0	0.06	
unit	m^2	m^2	m^3/h	m^3/h	m^2	

after pumps 1. On the other hand, tanks 2 and 3 have, as input flowrates, $\gamma_b q_b$ and $(1-\gamma_b)q_b$, respectively, where $\gamma_b \in [0,1]$ is the equivalent aperture ratio of the valve after pump 2. Tanks 3 and 4 discharge water into tanks 1 and 2, respectively. The parameters of the plant are given in Table I. The measurement noise is such that $|d_i(k)| \leq \bar{d}_i = 0.005$ for $i = 1,2$.

We define $x = [h_1, h_2, h_3, h_4]^T$, $u = [q_a, q_b]^T$, $y = [h_1, h_2]^T$. The sample time considered is $T_s = 60$ s. The input constraints are $q_a \in [0.1, 2.6]$ m^3/h and $q_b \in [0.4, 2.9]$ m^3/h, while the outputs must lie within the intervals $h_1 \in [0.1016, 1.1016]$ m and $h_2 \in [0.1097, 1.1097]$ m.

The MPC controller has a control horizon $\bar{p} = 5$. The Luenberger observer gain L is selected as the Kalman filter stationary gain while the auxiliary control law gain K is selected as a LQ gain, both of them obtained solving the discrete-time algebraic Riccati equations of optimal control theory and with diagonal matrices:

$$Q = \begin{bmatrix} \gamma_x I_{on_y} & 0_{on_y,(o-1)n_u} \\ 0_{(o-1)n_u,on_y} & \gamma_u I_{(o-1)n_u} \end{bmatrix}, R = \gamma_u I_{n_y}$$

where $\gamma_x = 0.25$ and $\gamma_u = 0.1$ for the observer and $\gamma_x = 1$ and $\gamma_u = 0.1$ for the auxiliary control law.

B. Application of the control scheme to nonlinear MIMO systems

The method discussed in the paper has been applied on the nonlinear system, using linear models identified from data generated by the nonlinear simulator (26). Since the control algorithm has been conceived with a focus on linear systems, a short discussion is due before to show the numerical results.

1) Identification and control issues in case of nonlinear systems: Using data generated by the nonlinear simulator (26), we have identified linear prediction models (2) and (3), together with the corresponding perturbation bounds $\bar{w}_i$, $i = 1,2$. However, the main assumptions guaranteeing the soundness of the learning phase (see [22]) require that the used model class includes the model of the system generating the data: this assumption is clearly impossible to be verified in this setting. In fact, the set of regressors used in (21) does not include nonlinear functions of input and state variables. In our opinion, however, this has not caused any significant problem in the considered case study which, remarkably, does not display a complex nonlinear dynamics (e.g. multiple equilibria, limit cycles, chaotic behaviour). The mismatch between the linear model and the nonlinear system has indeed been accurately included thanks to the disturbance term $w(k)$, leading to satisfactory simulation results, especially as far as the constraints fulfillment is regarded.

However, in the control design phase a different problem has arised from the fact that the nonlinear static gain is not constant, contrarily to the linear case. This problem has been here addressed by modifying the cost function (11), and in particular the final additive term $\sigma \|z_{\text{ref}}(k) - z_{\text{goal}}\|^2$. The idea used here consists of replacing z_{goal} with $\hat{\mu}\mu_{NL}(z_{\text{goal}})^{-1} z_{\text{goal}}$, where $\mu_{NL}(z_{\text{goal}})$ is the nonlinear system input-output static gain, computed on the working conditions defined by z_{goal}.

2) Numerical results: Considering the nonlinear case, Figure 1 shows the trends of the bounds against the prediction horizon.

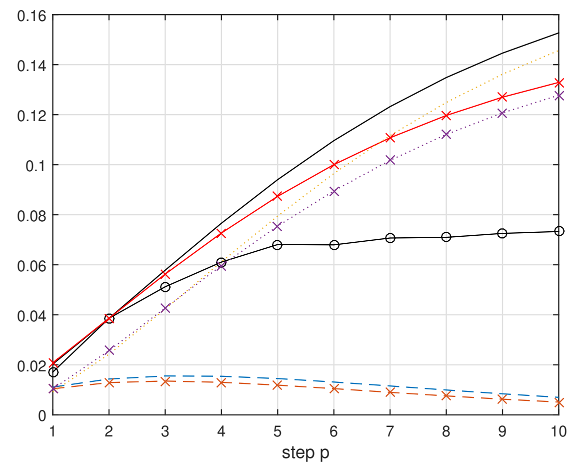

Fig. 1. Nonlinear case. Lines with marker "x" refer to output 2, while lines without a marker refer to output 1. Dashed lines: term $\sum_{i=1}^{on_y} |C_{j\bullet} A^p E_{\bullet i}| \bar{d}_{rem^*(i/n_y)}$ accounting for the wrong initialization of the state containing y samples instead of z ones, dotted lines: term$\sum_{i=0}^{p-1} \sum_{h=1}^{n_y} |C_{j\bullet} A^i M_{\bullet h}| \bar{w}_h$ accounting for the disturbance $\bar{w}$ integrated over time, line with circles: $\hat{\tau}_j^{(p)}(\hat{\Theta}^{*(1,p)})$, $j = 1,2$, solid lines: overall error bound $\sum_{i=0}^{p-1} \sum_{h=1}^{n_y} |C_{j\bullet} A^i M_{\bullet h}| \bar{w}_h + \sum_{i=1}^{on_y} |C_{j\bullet} A^p E_{\bullet i}| \bar{d}_{rem^*(i/n_y)}$

Figure 2 shows the closed-loop trajectories with the nonlinear model. Thanks to the modification of the final goal in the terminal cost, as described in this section, the output signals are able to track the desired references, reducing the steady state error, without harming the guaranteed theoretical properties of the controller. This fact can be appreciated thanks to Figure 3, which shows a detailed comparison between the control scheme including (solid black) or not including (dashed-dotted red) such modification, using the same disturbance signals.

VI. CONCLUSIONS AND FUTURE WORKS

A learning-based approach for robust predictive control for MIMO systems has been proposed and successfully tested on a benchmark example. The identification algorithm and the resulting controller are endowed with theoretical properties in the linear case, though they proved to be effective also on the nonlinear simulator. Preliminary extensions to address the system nonlinearities have been introduced, improving

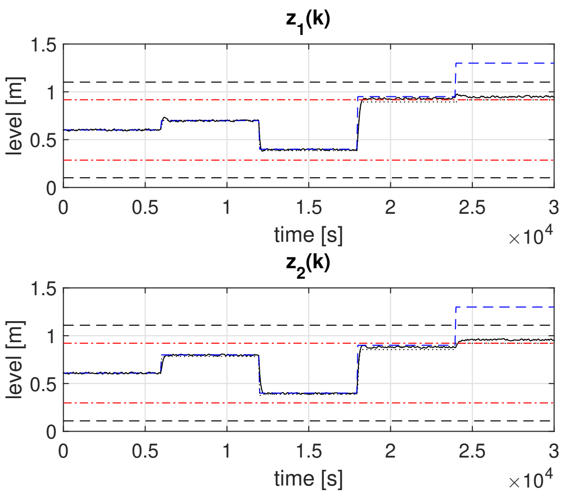

Fig. 2. Outputs from the closed-loop simulation with the nonlinear simulation. Solid black: output, dashed black: output constraints, dashed-dotted red: output tightened constraints, dashed blue: reference, dotted black: feasible reference.

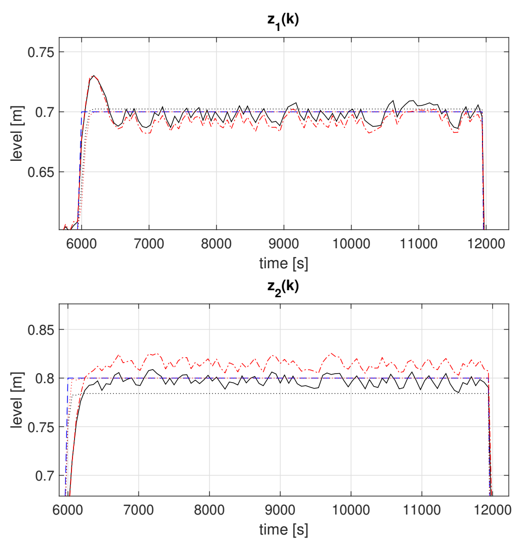

Fig. 3. Outputs from the closed loop-simulation with the nonlinear simulation: focus on a steady state period.

the static performance of the scheme. Future work includes the derivation of theoretical properties in the nonlinear framework, the analysis of the identification algorithm with models that are nonlinear in the regressors, and a testing phase on the real plant. Also, the extension to non-square systems is envisaged.

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