Preprint of final paper. DOI: 10.1002/asjc.560 © 2012 John Wiley and Sons Asia Pte Ltd and Chinese Automatic Control Society

Asian Journal of Control, Vol. 00, No. 0, pp. 1–9, Month 2008
Published online in Wiley InterScience (www.interscience.wiley.com) DOI: 10.1002/asjc.0000

DESIGN OF ROBUST PREDICTIVE CONTROL LAWS USING SET MEMBERSHIP IDENTIFIED MODELS†

M. Canale, L. Fagiano and M.C. Signorile

ABSTRACT

This paper investigates the robust design of Nonlinear Model Predictive Control (NMPC) laws that employ approximated models, derived directly from process input-output data. In particular, a Nonlinear Set Membership (NSM) identification technique is used to obtain a system model and a bound of the related uncertainty. The latter is used to carry out a robust control design, via a min-max formulation of the optimal control problem underlying the NMPC methodology. A numerical example with a nonlinear oscillator shows the effectiveness of the proposed approach.

Key Words: Predictive Control, Robust Stability, Nonlinear Control

I. INTRODUCTION

Nonlinear Model Predictive Control (MPC, see e.g. [14]), also referred to as receding horizon control, is a control technique in which the current control move is computed by solving on-line a constrained Finite Horizon Optimal Control Problem (FHOCP). In each sampling period, a measure or estimate of the system state is used as initial condition for the FHOCP and, according to a Receding Horizon (RH) strategy, only the first element of the solution sequence is applied to the system. Then, the procedure is repeated in the following sampling period, when a new measure of the state is available. The model employed in the FHOCP is typically a "physical" model (i.e. derived from physical laws) or a nonlinear parametric function (e.g. a neural network), whose parameters are identified by using measured process data.

In regard to the robustness analysis and robust design of NMPC, much progress has been made but many questions, such as uncertainty description and efficiency of the on-line computation, remain still open. [17] and [1] depicted the development of robust MPC, describing the several solutions proposed during the years. Using the contraction principle, [19] derived some necessary and sufficient conditions for robust stability, but they could result to be conservative and difficult to verify. [7] assured robust stability through the computation of some weights. Unfortunately, the existence of such weights was only a sufficient condition and consequently could be restrictive. [11] introduced a procedure to guarantee robust stability providing a non-restrictive result, which may turn out to be unsuitable for on-line computation because of its complexity. [20] proposed to achieve robust stability by enforcing a robust state contraction constraint through the optimization of a quadratic problem of medium size. The problem of designing predictive controllers in the presence of unmodeled dynamics was studied by [4] and [12]. More recently, [13] carried out a regional Input-to-State Stability (ISS) analysis of NMPC, [10] derived a suboptimal NMPC law with ISS guarantees, and [18] presented a robust NMPC scheme in the presence of state-dependent uncertainties and additive bounded perturbations. The concept of ISS has also been successfully exploited in [5] where a min-max MPC design approach has been introduced for the case

The authors are with the Dipartimento di Automatica e Informatica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Turin, Italy,(e-mail: {massimo.canale, lorenzo.fagiano,maria.signorile}@polito.it).
Lorenzo Fagiano is also with the Department of Mechanical Engineering, University of California, Santa Barbara - CA, USA.

This research has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n. PIOF-GA-2009-252284 - Marie Curie project "Innovative Control, Identification and Estimation Methodologies for Sustainable Energy Technologies."

of nonlinear time varying systems in the presence of delays.
Although several methods, like those described above, face the problem of robust stability, it has to be noted that, to the best of the authors' knowledge, in the nonlinear case there is no rigorous procedure to obtain a suitable description of the uncertainty associated with the employed model. This issue hampers the possibility to perform a systematic robustness analysis or a synthesis procedure to derive robust NMPC control laws. In fact, in most practical cases only a model of the system to be controlled is available, without any uncertainty description and/or estimate. Basically, this issue is due to the difficulty to evaluate model uncertainty when nonlinear parametric models, either "physical" or "black–box", are employed. Indeed, the parameters of such models are usually identified from system input/output data. With such a procedure it is not easy, in general, to derive also some uncertainty estimate to be used for robustness analysis or robust control design.
In order to face these issues, this paper proposes an approach, named Set Membership Predictive Control (SMPC), to design a predictive control law directly from measured input/output data. In SMPC, a Nonlinear Set Membership (NSM) identification methodology (see [15] for details), able to obtain both a "nominal" system model and a bound on the related uncertainty, is used. Such a bound can then be employed to analyze a posteriori the robustness of a NMPC law designed for the nominal model, as it has been done by [3], or to design a priori a robust predictive controller for the system. The latter option is pursued in this work, by using a min-max formulation of the FHOCP. Finally, as an example, the SMPC approach is applied here to a nonlinear oscillator.

II. NONLINEAR SET MEMBERSHIP MODELS FOR NMPC

NMPC requires the on–line solution of a constrained FHOCP, in which the predicted system behavior is computed using a model. To this end, existing NMPC approaches use nonlinear models such as "physical" models (i.e. derived from physical laws) or nonlinear parametric functions (e.g. a neural network). The novelty introduced in this work is that the model embedded in the NMPC algorithm is identified directly from measured input-output data, using a NSM technique. Such methodology allows to obtain an uncertainty bound useful for the robust control design, as well as an "optimal" nominal model, i.e. a model with minimal uncertainty bound w.r.t. the information of the available data.
This Section presents details on the NSM approach and its use to derive a model suitable for robust NMPC design.

2.1. Nonlinear Set Membership identification

The NSM identification approach of this work is derived from the methodology proposed in [15].
Suppose that the plant $\mathbf{P}$ to be controlled is a single-input, single-output nonlinear discrete-time dynamic system described in regression form:

$$\begin{aligned} y_{t+1} &= P(\mathbf{y}_t, \mathbf{u}_t) \qquad t \in \mathbb{Z} \\ \mathbf{y}_t &= [y_t; \ldots; y_{t-n_y}] \\ \mathbf{u}_t &= [u_t; \ldots; u_{t-n_u}] \end{aligned} \tag{1}$$

where $u_t, y_t \in \mathbb{R}$, $P : \mathbb{R}^n \to \mathbb{R}$, $n = n_y + n_u + 2$.
Suppose that system $\mathbf{P}$ is not known, but a set of noise-corrupted measurements is available

$$(\widetilde{y}_t, \widetilde{u}_t) \quad t \in \mathcal{T} \doteq \{-T+1, -T+2, ..., 0\} \tag{2}$$

Let $\widetilde{\varphi}_t \doteq [\widetilde{\mathbf{y}}_t; \widetilde{\mathbf{u}}_t]$ where $\widetilde{\mathbf{y}}_t = [\widetilde{y}_{t-1}; \ldots; \widetilde{y}_{t-n_y}]$ and $\widetilde{\mathbf{u}}_t = [\widetilde{u}_t; \ldots; \widetilde{u}_{t-n_u}]$. Then, P can be rewritten as

$$\widetilde{y}_{t+1} = P(\widetilde{\varphi}_t) + s_t, \; t \in \mathcal{T} \tag{3}$$

where the term s_t accounts for the fact that y_t and φ^t are not exactly known.
The aim is to derive a model M of P from the available measurements $(\widetilde{y}_t, \widetilde{u}_t)$. The estimate M should be chosen to give small (possibly minimal) L_p error $||P - M||_p$, where the symbol $|| \cdot ||_p$ denotes the spatial $p-$norm of a given function $F(\varphi)$ of the variable $\varphi \in \mathbb{R}^n$ defined as $\|F\|_p \doteq \left[\int_\phi |F(\varphi)|^p \, d\varphi\right]^{\frac{1}{p}}, p \in (1, \infty)$, $\|F\|_\infty \doteq \operatorname{ess\,sup}_{\varphi \in \Phi} |F(\varphi)|$, $| \cdot |$ is the Euclidean norm and Φ is a bounded set in $\mathbb{R}^n$ which belongs to the domain of $F(\cdot)$.
Whatever estimate is chosen, no information on the identification error can be derived, unless some assumptions are made on the function P and the noise s.
Prior assumptions on P: $\quad P \in \mathcal{F}(\gamma)$

$$\begin{aligned} \mathcal{F}(\gamma) \doteq \{F \in C^0 : |F(\varphi) - F(\bar{\varphi})| \le \gamma |\varphi - \bar{\varphi}|, \\ \forall \varphi, \bar{\varphi} \in \Phi \subset \mathbb{R}^n\} \end{aligned} \tag{4}$$

Prior assumptions on noise:

$$|s_t| \le \varepsilon < \infty, \; t \in \mathcal{T}. \tag{5}$$

Thus, $\mathcal{F}(\gamma)$ is the set of Lipschitz continuous functions on Φ with Lipschitz constant γ. It is assumed that Φ is a compact set.
A key role in this Set Membership framework is played by the Feasible Systems Set, often called "unfalsified systems set", i.e. the set of all systems consistent with prior information and measured data.

Definition 1 *Feasible Systems Set:*

$$FSS \doteq \{F \in \mathcal{F}(\gamma) \colon |\widetilde{y}_t - F(\widetilde{\varphi}_t)| \leq \varepsilon, t \in \mathcal{T}\}. \quad (6)$$

■

The Feasible Systems Set FSS summarizes all the information on the mechanism generating the data that is available up to time $t = 0$. If prior assumptions are "true", then $P \in FSS$. Indeed, for a given estimate $M \simeq P$, the related L_p error $\|P - M\|_p$ cannot be exactly computed, since P is not known, but its tightest bound is given by

$$\|P - M\|_p \leq \sup_{F \in FSS} \|F - M\|_p$$

This motivates the following definition of worst-case identification error.

Definition 2 *The worst-case identification error of the estimate M is*

$$E(M) \doteq \sup_{F \in FSS} \|F - M\|_p .$$

■

Looking for estimates that minimize the worst-case identification error leads to the following optimality concept.

Definition 3 *An estimate F^* is optimal if*

$$E(F^*) = \inf_M E(M) = \mathcal{R}_{I,p}.$$

■

The quantity $\mathcal{R}_{I,p}$, called *radius of information*, gives the minimal worst-case identification error that can be guaranteed by any estimate based on the available information.
Define the functions:

$$\begin{aligned} \overline{F}(\varphi) &\doteq \min_{t \in \mathcal{T}} \left(\overline{h}_t + \gamma|\varphi - \widetilde{\varphi}_t|\right) \\ \underline{F}(\varphi) &\doteq \max_{t \in \mathcal{T}} \left(\underline{h}_t - \gamma|\varphi - \widetilde{\varphi}_t|\right) \end{aligned} \quad (7)$$

where $\overline{h}_t \doteq \widetilde{y}_t + \varepsilon$, $\underline{h}^t \doteq \widetilde{y}_t - \varepsilon$. The next result shows that the estimate:

$$M_c \doteq \frac{1}{2}\left(\underline{F} + \overline{F}\right) \quad (8)$$

is optimal for any L_p norm.

Theorem 1 Theorem 7 of [15].
For any L_p norm, with $p \in [1, \infty]$:
i) The estimate M_c is optimal.
ii) $E(M_c) = \frac{1}{2}\left\|\overline{F} - \underline{F}\right\|_p = \mathcal{R}_{I,p}$.
iii) For any $\phi \in \Phi$
$|M_c(\phi) - P(\phi)| \leq \dfrac{1}{2}|\overline{F}(\phi) - \underline{F}(\phi)|$.

■

Note that the model M_c can be expressed as a nonlinear regression of the form:

$$y_{t+1} = M_c(y_t; \ldots; y_{t-n_y}, u_t; \ldots; u_{t-n_u}) \qquad t \in \mathbb{Z} \quad (9)$$

where M_c is a Lipschitz continuous function with Lipschitz constant γ (see [15]).

2.2. Pseudo-state representation of NSM models and uncertainty description

For the SMPC approach employed in this paper a state space representation of (1) and (9) is needed. In particular, the regression (1) can be easily represented in the context of state space equations. In fact, by choosing a "pseudo-state" vector as:

$$\begin{aligned} x_t &= [y_t \ldots y_{t-n_y} u_{t-1} \ldots u_{t-n_u}]^T = \\ &= [x_t^{(1)} \ldots x_t^{(n_y+1)} x_t^{(n_y+2)} \ldots x_t^{(n_y+n_u+1)}]^T \end{aligned} \quad (10)$$

and as input the value u_t, the regression form (1) can be expressed as

$$x_{t+1} = f^P(x_t, u_t) \quad (11)$$

where:

$$f^P(x_t, u_t) =$$

$$\begin{bmatrix} P(x_t^{(1)}, \ldots, x_t^{(n_y+1)}, u_t, x_t^{(n_y+2)}, \ldots, x_t^{(n_y+n_u+1)}) \\ x_t^{(1)} \\ \vdots \\ x_t^{(n_y)} \\ u_t \\ \vdots \\ x_t^{(n_y+n_u)} \end{bmatrix} \quad (12)$$

Note that, since $P(\cdot)$ is assumed to be Lipschitz continuous with constant γ, function $f^P(\cdot)$ in (12) is Lipschitz continuous too with constant $L_P = \sqrt{1+\gamma^2}$.
The same procedure, applied to the model M_c (9), leads to the state space description:

$$x_{t+1} = f^{M_c}(x_t, u_t) \quad (13)$$

where:

$$f^{M_c}(x_t,u_t) = \begin{bmatrix} M_c(x_t^{(1)},\ldots,x_t^{(n_y+1)},u_t,x_t^{(n_y+2)}\ldots,x_t^{(n_y+n_u+1)}) \\ x_t^{(1)} \\ \vdots \\ x_t^{(n_y)} \\ u_t \\ \vdots \\ x_t^{(n_y+n_u)} \end{bmatrix} \quad (14)$$

Indeed, based on the assumptions made in the considered NSM identification setup, function $f^{M_c}(\cdot)$ in (13) is Lipschitz continuous with the same constant of $f^P(\cdot)$ $L_P = \sqrt{1+\gamma^2}$. Moreover, the NSM identification procedure returns an estimate of the uncertainty associated with the model (13), too. In fact, with a slight abuse of notation, from (12) and (14), it can be obtained that

$$\begin{aligned} f^{M_c}(x_t,u_t) \; &- \; f^P(x_t,u_t) = \\ &= \begin{bmatrix} M_c(x_t,u_t) - P(x_t,u_t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &\doteq \; w_t(x_t,u_t) \end{aligned} \quad (15)$$

thus the following equation can be derived:

$$x_{t+1} = f^P(x_t,u_t) = f^{M_c}(x_t,u_t) + w_t(x_t,u_t) \quad (16)$$

i.e. the model uncertainty is described in terms of an additive, input-and-state-dependent perturbation $w_t(x_t,u_t) \in \mathbb{R}^{n_y+n_u+1}$, whose elements are all equal to zero except for the first one. Then, according to Theorem 1-iii), it can be shown that the quantity $w_t(x_t,u_t)$ is pointwise bounded as:

$$\begin{aligned} |w_t(x_t,u_t)| &\le \frac{1}{2}|\overline{F}(x_t,u_t) - \underline{F}(x_t,u_t)| \\ &\doteq \overline{w}(x_t,u_t), \text{ for any } (x_t,u_t) \in \Phi, \end{aligned} \quad (17)$$

where the bound $\overline{w}(x_t,u_t)$ is computed in the NSM approach together with the estimate $M_c(x_t,u_t)$. Note that, in (17), $w_t \in \mathbb{R}^{n_y+n_u+1}$, while $\overline{F}(x_t,u_t)$, $\underline{F}(x_t,u_t) \in \mathbb{R}$: yet, eq. (17) holds due to the particular structure of w_t (15).

In principle, one could use the nominal model (13) and the uncertainty bound (17) to design a robust controller for the system (11). However, the related control design may be too complex. Thus, in this paper a "global", rather than pointwise, uncertainty bound is employed, derived by using Theorem 1-ii) with the ∞-norm as a measure of accuracy of the estimate M w.r.t. the real system:

$$\begin{aligned} \forall (x,u) \in \Phi,\; \overline{w}(x,u) &= \frac{1}{2}|\overline{F}(x,u) - \underline{F}(x,u)| \\ &\le \frac{1}{2}\|\overline{F}(x,u) - \underline{F}(x,u)\|_\infty = \mathcal{R}_{I,\infty} \doteq \mu. \end{aligned} \quad (18)$$

Summing up, on the basis of eqs. (16)-(18) the pseudo-state model to be employed in the robust NMPC design is the following:

$$x_{t+1} = f^{M_c}(x_t,u_t) + w_t,\; |w_t| \le \mu, \quad (19)$$

where x_t, $w_t \in \mathbb{R}^{n_u+n_y+1} = \mathbb{R}^r$ and $u_t \in \mathbb{R}$. An estimate of the bound μ can be computed e.g. by using the approach of [16].

III. ROBUST CONTROL DESIGN FOR SMPC

In this Section, the robust design of a SMPC control law is described.

In the following, the sequence of $\overline{k}$ control inputs $\{u_t\}_{t_1}^{t_1+\overline{k}-1}$, starting from the generic time instant $t = t_1$ up to time instant $t = t_1 + \overline{k} - 1$, is indicated as $U_{t_1}^{\overline{k}}$. Similarly, $W_{t_1}^{\overline{k}}$ indicates a sequence $\{w_t\}_{t_1}^{t_1+\overline{k}-1}$ of "disturbances" from time instant t_1 up to time instant $t_1 + \overline{k} - 1$. The set of all the possible state values at time $t_1 + \overline{k}$, that originate from the generic state value x_{t_1} at time t_1 by applying the input sequence $U_{t_1}^{\overline{k}}$ to system (19), is defined as:

$$\begin{aligned} \mathcal{S}(x_{t_1},U_{t_1}^{\overline{k}}) \; &\doteq \; \{\{x_t\}_{t_1}^{t_1+\overline{k}} : \\ x_{t_1+k+1} \; &= \; f^{M_c}(x_{t_1+k},u_{t_1+k}) + w_{t_1+k}, \\ \forall k \in [0,\overline{k}-1] \quad &, \quad \forall w_{t_1+k} : |w_{t_1+k}| \le \mu\} \end{aligned} \quad (20)$$

while $\phi(x_{t_1},U_{t_1}^{\overline{k}})$ indicates the nominal state value (i.e. with $w_t = 0\,\forall t$) and $\phi^P(x_{t_1},U_{t_1}^{\overline{k}})$ is the state value of the real plant at time $t_1+\overline{k}$, obtained starting at x_{t_1} and applying the input sequence $U_{t_1}^{\overline{k}}$. Clearly, it holds that $\{\phi(x_{t_1},U_{t_1}^{\overline{k}}), \phi^P(x_{t_1},U_{t_1}^{\overline{k}})\} \subset \mathcal{S}(x_{t_1},U_{t_1}^{\overline{k}})$. The Hausdorff distance (see e.g. [2]) between any two sets $\mathcal{S} \in \mathbb{R}^r$ and $\mathcal{X} \in \mathbb{R}^r$ is defined as:

$$d(\mathcal{S},\mathcal{X}) = \max\left(\sup_{x^1\in\mathcal{S}} \inf_{x^2\in\mathcal{X}} |x^1 - x^2|, \sup_{x^1\in\mathcal{X}} \inf_{x^2\in\mathcal{S}} |x^1 - x^2| \right) \quad (21)$$

It is assumed that the control problem is to robustly asymptotically regulate the state of system (19) to a convex and compact neighborhood of the origin, indicated as $\mathcal{X}_f \subseteq \mathbb{R}^r$, under state and input constraints, indicated respectively by a convex set $\mathbb{X} \subseteq \mathbb{R}^r$ and a convex, compact set $\mathbb{U} \subseteq \mathbb{R}$, both containing the origin in their interiors. The notation $U_{t_1}^{\overline{k}} \in \mathbb{U}$ indicates that each one of the elements of the sequence $U_{t_1}^{\overline{k}}$ belongs to $\mathbb{U}$. The following assumption is considered for $\mathcal{X}_f$:

Assumption 1
$\forall x_t \in \mathcal{X}_f,\, \forall \overline{k} \in [1, \infty),\, \exists U_t^{\overline{k}} \in \mathbb{U} : \mathcal{S}(x, U_t^{\overline{k}}) \in \mathcal{X}_f,$

i.e. there exists a feasible control sequence that robustly keeps the state inside the set $\mathcal{X}_f$ for any future time step. The set $\mathcal{X}_f \subset \mathbb{X}$ is a design parameter that has to be chosen according to a tradeoff between better regulation precision and NMPC problem feasibility, as it will be clear in the following. Obviously, $\mathcal{X}_f$ can not be chosen arbitrarily small, due to the presence of the uncertainty w. By indicating as $N \in \mathbb{N}$ the prediction horizon, the following cost function J can be defined:

$$J(x_t, U_t^N) = d(x_t, \mathcal{X}_f) + \sum_{k=1}^{N-1} d(\mathcal{S}(x_t, U_t^k), \mathcal{X}_f) \quad (22)$$

then, the FHOCP to be solved in the SMPC approach is:

$$J^*(x_t) = \min_{U_t^N} J(x_t, U_t^N) \quad (23a)$$

subject to

$$\mathcal{S}(x, U_t^k) \in \mathbb{X},\, \forall k \in [1, N] \quad (23b)$$

$$U_t^N \in \mathbb{U} \quad (23c)$$

$$\mathcal{S}(x, U_t^N) \in \mathcal{X}_f \quad (23d)$$

A (possibly local) optimal control sequence is indicated as $U_t^{N*}(x_t)$. The following assumption is considered about the constrained FHOCP (23):

Assumption 2 *There exists a set $\mathcal{F} \in \mathbb{R}^r$ such that the FHOCP (23) is feasible $\forall x \in \mathcal{F}$.*

Remark 1 *The feasibility of (23) depends on many factors, such as the model (19), the related uncertainty bound, the constraint sets and the choice of the terminal set $\mathcal{X}_f$. In the quite general settings of this paper, it is difficult to derive sufficient conditions on these factors to satisfy Assumption 2 and to evaluate the feasibility set $\mathcal{F} \in \mathbb{R}^r$. These aspects are beyond the scope of this paper and are subjects of future research.*

The FHOCP (23) is typically solved numerically. In particular, it is assumed that the employed algorithm, denoted as $U_t^{N*}(x_t) = \lambda(x_t)$, enjoys the following properties:

Assumption 3 *For any $x \in \mathcal{F}$, $\lambda(x)$ returns a (eventually local) minimum $J^*(x)$ and the related minimizer $U_t^{N*}(x)$*

Assumption 4

(a) *For any predicted time instant $t + \underline{k}$ and any control sequence $U_t^{\underline{k}} \in \mathbb{U}$ such that $\mathcal{S}(x_t, U_t^{\underline{k}}) \in \mathcal{X}_f$, the algorithm $\lambda(x_t)$ is able to compute a control sequence $\hat{U}_{t+\underline{k}}^{N-\underline{k}} \in \mathbb{U}$, such that $\mathcal{S}(\mathcal{S}(x_t, U_t^{\underline{k}}), \hat{U}_{t+\underline{k}}^{k}) \in \mathcal{X}_f,\, \forall k \in [1, N - \underline{k}]$.*

(b) *The minimizer $U_t^{N*}(x_t)$ provided by $\lambda(x_t)$ is such that, for any predicted time instant $t + \underline{k} : \mathcal{S}(x_t, U_t^{\underline{k}*}) \in \mathcal{X}_f,\, \underline{k} \in [0, N]$, it happens that $\mathcal{S}(x_t, U_t^{\underline{k}+k*}) \in \mathcal{X}_f,\, \forall k \in [1, N - \underline{k}]$, i.e. the state trajectories are robustly kept inside the terminal set $\mathcal{X}_f$.*

Remark 2 *Assumptions 3-4 are quite mild, provided that Assumptions 1-2 hold. In particular, with the settings of this paper Assumption 3 is satisfied if the problem is feasible and the solver is initialized with a feasible solution (or is able to find a feasible solution). Assumption 4(a) can be satisfied if Assumption 1 holds (so that there exists a sequence that robustly keeps the state inside the terminal set) and if the algorithm $\lambda(x_t)$ is able to find a sequence $\hat{U}_{t+\underline{k}}^{N-\underline{k}}$ and a scalar $\hat{t} = 0$ that solve the following optimization problem:*

$$\left(\hat{t},\, \hat{U}_{t+\underline{k}}^{N-\underline{k}}\right) = \arg \min_{t, U_{t+\underline{k}}^{N-\underline{k}}} t$$

s.t.

$$\mathcal{S}(\mathcal{S}(x_t, U_t^{\underline{k}}), \hat{U}_{t+\underline{k}}^{k}) \le t,\, \forall k \in [1, N - \underline{k}].$$

Finally, it can be noted that Assumption 4(b) holds as a consequence of Assumption 4(a), by considering that, according to the chosen cost function (22), the stage cost related to any predicted set $\mathcal{S}(x_t, U_t^k) : \mathcal{S}(x_t, U_t^k) \in \mathcal{X}_f$ is zero (i.e. minimal).
Assumptions 4(a)-(b) can be replaced by assuming that a terminal control policy is known, under which the set $\mathcal{X}_f$ is robustly positively invariant (see e.g. [14]). In some sense, in this paper the terminal control policy is not known a priori, while it is assumed that the algorithm $\lambda(x_t)$ is able to derive it.

According to the RH strategy, the SMPC controller is implemented as follows:

Algorithm 1

1. *At time instant t, get x_t.*
2. *Solve (23), by initializing the algorithm $\lambda(x_t)$ with the optimal sequence $\tilde{U}_t^{N*}$, computed at time instant $t-1$ and suitably shifted.*
3. *Apply the first element of the solution sequence U_t^{N*} as the actual control action u_t.*
4. *Repeat the whole procedure at time $t+1$.*

The control law resulting from Algorithm 1 is indicated here as $u_t = \kappa^*(x_t)$, and the related sequence, starting from the generic time instant t_1 up to time t_1+k-1, is denoted as $K_{t_1}^k = \{\kappa^*(x_t)\}_{t_1}^{t_1+k-1}$. The following stability result holds.

Theorem 2 *Under Assumptions 1-4, the distance between the state of system (19), controlled by the feedback law κ^*, and the terminal set $\mathcal{X}_f$ asymptotically robustly converges to zero for any initial condition $x_t \in \mathcal{F}$, i.e.:*

$$\forall x_t \in \mathcal{F},\ \lim_{k\to\infty} d(\mathcal{S}(x_t, K_t^k), \mathcal{X}_f) = 0$$

Proof 1 *See the appendix.*

Remark 3 *Theorem 2 also implies that the distance between the state $\phi^P(x_{t_1}, U_{t_1}^{\bar{k}})$ of the controlled system and the set $\mathcal{X}_f$ asymptotically converges to zero.*

Remark 4 *In order to reduce the conservativeness of the presented approach and to improve the feasibility of (23), the FHOCP can be generalized by optimizing over control policies κ, i.e. $U_{t_1}^N = \{\kappa(x_t)\}_{t_1}^{t+N-1} + V_{t_1}^{t+N-1}$, so that the predictions involved in (23) can be carried out in a closed-loop fashion. It is widely recognized (see e.g. [8]) that this approach leads to better performance and reduced feasibility problems. Optimization over control policies has not been adopted in the theoretical framework of this paper just for simplicity of notation, yet it can be straightforwardly used and indeed it has been employed in the numerical example of Section IV. Another approach that can be successfully exploited for the special case of linear systems, involves the use of parameter dependent open loop optimization as introduced in [6].*

Remark 5 *It has to be noted that the set $\mathcal{F}$ has to be a subset (with lower dimension) of the set Φ, over which the NSM identification procedure is applied.*

IV. NUMERICAL EXAMPLE

Consider the following two-dimensional, discrete-time nonlinear oscillator obtained from the Duffing equation (see e.g. [9]):

$$\begin{aligned} \xi_{t+1} &= \begin{bmatrix} 1 & T_s \\ -T_s\,\omega^2 & 1-2\,\zeta\,T_s \end{bmatrix} \xi_t + \\ &\quad \begin{bmatrix} 0 & 0 \\ -T_s & 0 \end{bmatrix} \xi_t^3 + \begin{bmatrix} 0 \\ T_s \end{bmatrix} u_t \\ y_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \xi_t + v_t \end{aligned} \tag{24}$$

where $\xi_t = [\xi_t^{(1)}\ \ \xi_t^{(2)}]^T$ is the system state (the symbol $\cdot^T$ denotes the transpose operator), $v_t \in [-0.01, 0.01]$ is an unknown-but-bounded measurement noise, $\zeta = 0.3$, $\omega = 1$ and $T_s = 0.05$ s.
The control objective is to regulate the output y to the origin, under the following output and input constraints:

$$|y| \leq 3 \qquad |u| \leq 5 \tag{25}$$

The system (24) is supposed to be unknown, but a set of noise-corrupted measurements can be collected through preliminary experiments. Note that the origin of system (24) is an open-loop asymptotically stable fixed point for any initial condition $\xi_0 \in \mathbb{R}^2$, so that the preliminary experiment can be carried out in open-loop fashion. In particular, 30 experiments have been carried out, starting from 30 different initial conditions $\xi_0 \in \mathbb{R}^2 : \|\xi_0\|_\infty \leq 3$. In each one of these experiments, a uniformly distributed random sequence $\{\tilde{u}_t\}_0^{1\cdot 10^3} : \forall t,\ |\tilde{u}_t| \leq 5$ has been used as input, and a second uniformly distributed random sequence $\{\tilde{v}_t\}_0^{1\cdot 10^3} : \forall t,\ |\tilde{v}_t| \leq 0.01$ has been employed as measurement noise. The overall collected data form a set of $3\cdot 10^4$ samples $(\tilde{y}, \tilde{u})$ (2), which has been split in an identification set of $2.5\cdot 10^4$ samples, to be used in the NSM identification procedure, and in a validation set containing the remaining $5\cdot 10^3$ samples. The number of output and input regressors, n_y and n_u respectively, have been chosen in order to achieve a suitable tradeoff between model complexity and accuracy, while the values of the Lipschitz constant γ and of the noise bound ε have been estimated from the data in order to achieve a non-empty FSS (for more details on the regressor's choice and on the computation of γ and ε, the interested reader is referred to [15]). In this case, the values $n_y = 2$, $n_u = 2$, $\gamma = 2.3$ and $\varepsilon = 0.02$ have been chosen. The obtained NSM model has the form (19):

$$x_{t+1} = f^{M_c}(x_t, u_t) + w_t$$

where

$$x_t = \begin{bmatrix} x_t^{(1)} \\ x_t^{(2)} \\ x_t^{(3)} \end{bmatrix} = \begin{bmatrix} y_t \\ y_{t-1} \\ u_{t-1} \end{bmatrix} \tag{26}$$

$$f^{M_c}(x_t, u_t) = \begin{bmatrix} M_c(x_t, u_t) \\ x_t^{(1)} \\ u_t \end{bmatrix} \tag{27}$$

and $M_c(x_t, u_t)$ is the identified NSM model (see (14)). The estimated uncertainty bound μ results to be equal to 0.1. The derived model is then employed to design the SMPC law, according to Algorithm 1, by optimizing over linear feedback control policies of the form $u_t = K\, x_t + u'_t$ (see Remark 4). In particular, the horizon $N = 30$ and the terminal set $\mathcal{X}_f = \{x \in \mathbb{R}^3 : |x^{(1)}|, |x^{(2)}| \leq 0.1; |x^{(3)}| \leq 2\}$ have been employed. Finally, on the basis of the constraints (25) on the actual system and of the pseudo-state choice (26), the sets $\mathbb{X}$ and $\mathbb{U}$ are selected as follows:

$$\begin{aligned} \mathbb{X} &= \left\{x \in \mathbb{R}^3 : |x^{(1)}|, |x^{(2)}| \leq 3;\ |x^{(3)}| \leq 5\right\} \\ \mathbb{U} &= \{u \in \mathbb{R} : |u| \leq 5\} \end{aligned}$$

The obtained results, starting as an example from the initial state $\xi_0 = [1.85, -3.41]^T$ (and initial pseudo-state $x_0 = [1.85,\ 2,\ 0]^T$), are shown in Fig. 1-3, where they are compared to the results achieved in an "ideal" case, i.e. with a NMPC law designed and implemented assuming exact knowledge of the system equations (24) and measurement of the whole state with zero noise. In particular, Figs. 1 and 2(a)-(b) show the trajectories of the system output y and state ξ respectively. It can be noted that quite good regulation precision is achieved by the SMPC law (see Fig. 2(b)), while its performance in the transient phase are worse w.r.t. the standard NMPC law, due to the conservativeness of the robust design employed in SMPC and the presence of measurement noise. The courses of the input u are shown in Fig. 3, where it can be noted that the input constraints are always satisfied by both controllers.

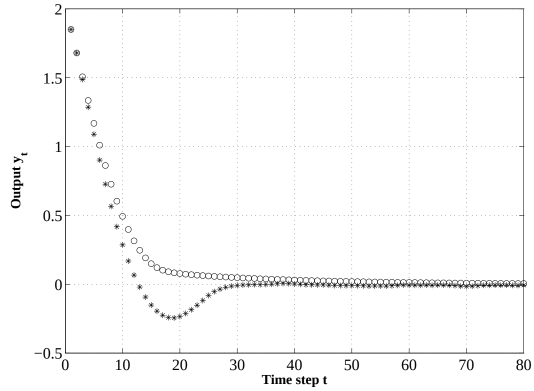

Fig. 1. Numerical example: courses of the system output y with the SMPC control law ($*$) and with a state-feedback NMPC law ($\circ$). Initial state: $\xi_0 = [1.85, -3.41]^T$, initial pseudo-state: $x_0 = [1.85,\ 2,\ 0]^T$.

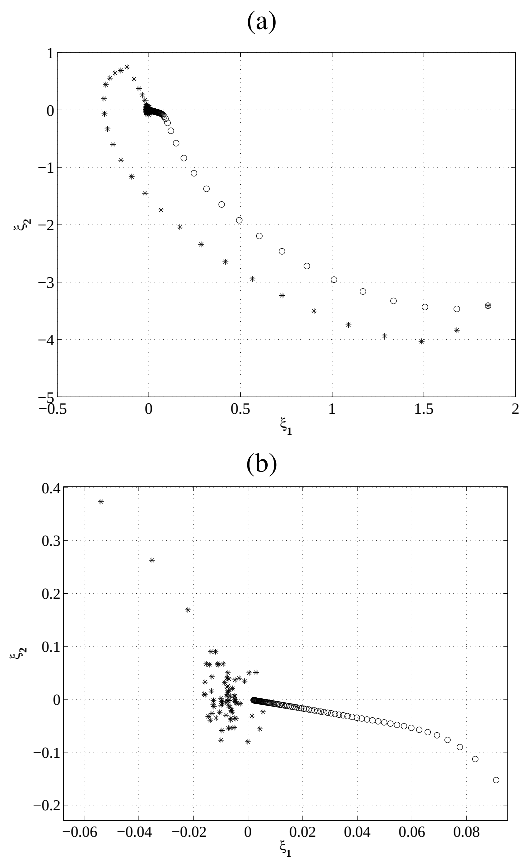

Fig. 2. Numerical example: (a) Trajectories of the system state ξ with the SMPC control law ($*$) and with a state-feedback NMPC law ($\circ$). (b) Zoom of the trajectories close to the origin. Initial state: $\xi_0 = [1.85, -3.41]^T$, initial pseudo-state: $x_0 = [1.85,\ 2,\ 0]^T$.

V. CONCLUSIONS

The robust design of Nonlinear Model Predictive Control laws that employ approximated models, derived directly from input-output data, has been studied in this paper. Such models are identified by means of a Nonlinear Set Membership identification technique which is able to provide an estimate of the uncertainty associated to the model. The obtained uncertainty bound is employed to design a robust predictive controller by using a min-max formulation of the finite horizon optimal control problem. The effectiveness of the approach has been shown through

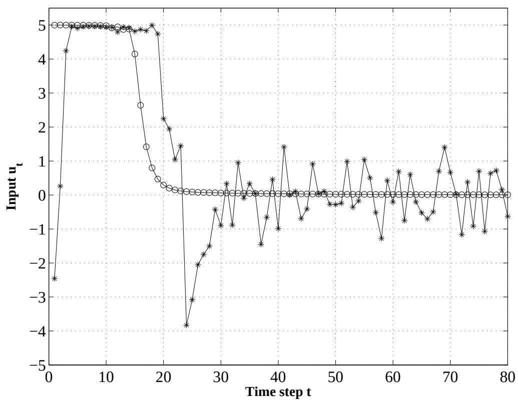

Fig. 3. Numerical example: courses of the system input u with the SMPC control law ($*$) and with a state-feedback NMPC law ($\circ$). Initial state: $\xi_0 = [1.85,\ -3.41]^T$, initial pseudo-state: $x_0 = [1.85,\ 2,\ 0]^T$.

a nonlinear oscillator example.

Appendix - Proof of Theorem 2

At first, the recursive feasibility of Algorithm 1 is analyzed. Take any $x_t \in \mathcal{F}$. Assumptions 2 and 3 imply that the algorithm $\lambda(x_t)$ is able to find a feasible, locally optimal solution sequence. Such a sequence is indicated here as $U^{N*}_{t|t} \in \mathbb{U}$, to highlight that it is the solution of (23) at time t. By applying the first element of such a sequence to the system, the state $x_{t+1} = \phi^P(x_t, U^{1*}_{t|t})$ is obtained at the following time step, $t+1$. By initializing the algorithm $\lambda(x_{t+1})$ with the shifted optimal sequence $U^{N-1*}_{t+1|t}$ computed at time step t, it can be noted that $S(x_{t+1}, U^{k*}_{t+1|t}) \in \mathbb{X}$, $\forall k \in [0, N-1)$ and that the set $S(x_{t+1}, U^{N-1*}_{t+1|t}) \in \mathcal{X}_f$, i.e. at the second last prediction step the state trajectory is robustly inside the terminal set $\mathcal{X}_f$. Then, according to Assumptions 1 and 4, the algorithm $\lambda(x_{t+1})$ is able to find a control input $\hat{u}_{t+N} \in \mathbb{U}$ so that $S(x_{t+1}, \hat{U}^{N*}_{t+1|t}) \in \mathcal{X}_f$, where the sequence $\hat{U}^{N*}_{t+1|t} \in \mathbb{U}$ is constructed by using as first $N-1$ components the elements $\{u^*_t\}^N_{t+1|t}$ of the optimal sequence $U^{N*}_{t|t}$, and as the last component the value $\hat{u}_{t+N}$. The sequence $\hat{U}^{N*}_{t+1|t}$ provides a feasible input sequence for the problem (23) at time x_{t+1}. Such a reasoning can be iterated for any time instant $t+k, k \in [2, \infty)$, so that recursive feasibility is proved. The asymptotic convergence of the distance $d(\mathcal{S}(x_t, K^k_t), \mathcal{X}_f)$ to zero, as $k \to \infty$, will be now proved. For any state value $x_t \in \mathcal{F}$, consider the optimal cost $J^*(x_t)$, computed at time t by algorithm $\lambda(x_t)$, corresponding to the optimal solution sequence $U^{N*}_{t|t}(x_t)$. For the sake of simplicity of notation, define $d(x_t) \doteq d(x_t, \mathcal{X}_f)$. From the definition of the cost function J (22), it can be noted that

$$0 \le d(x_t) \le J^*(x_t), \tag{28}$$

i.e. the distance between the state x_t and the set $\mathcal{X}_f$ is upper-bounded by $J^*(x_t)$. Moreover, due to Assumptions 1 and 4,

$$J^*(x_t) = 0 \iff d(x_t) = 0 \tag{29}$$

so that $J^*(x_t) = 0$ if and only if $x_t \in \mathcal{X}_f$. Finally, consider the difference $J^*(x_{t+1}) - J^*(x_t)$. Since the algorithm $\lambda(x_{t+1})$ at time $t+1$ is provided with the feasible input sequence $\hat{U}^{N*}_{t+1|t}$, which is suboptimal, it holds that:

$$J^*(x_{t+1}) \le J(x_{t+1}, \hat{U}^{N*}_{t+1|t}). \tag{30}$$

$J(x_{t+1}, \hat{U}^{N*}_{t+1|t})$ is such that:

$$J(x_{t+1}, \hat{U}^{N*}_{t+1|t}) \le J^*(x_t) - d(x_t) \tag{31}$$

By combining eqs. (30) and (31), it can be noted that:

$$J^*(x_{t+1}) - J^*(x_t) \le -d(x_t) \tag{32}$$

with:

$$J^*(x_{t+1}) - J^*(x_t) = 0 \iff d(x_t) = 0, \tag{33}$$

in which case $J^*(x_{t+1}) = J^*(x_t) = 0$, due to Assumption 4. Thus, it holds that:

$$\begin{array}{l} J^*(x_{t+1}) - J^*(x_t) < 0 \,\forall x_t \in \mathcal{F} \setminus X_f \\ J^*(x_{t+1}) - J^*(x_t) = 0 \iff x_t \in \mathcal{X}_f \end{array} \tag{34}$$

Equations (28) and (34) are sufficient to prove robust asymptotic convergence of $d(x_t)$ to 0:

$$\lim_{t\to\infty} d(x_t) \le \lim_{t\to\infty} J^*(x_t) = 0, \,\forall x_0 \in \mathcal{F} \tag{35}$$

REFERENCES

1. A. Bemporad and M. Morari. Robust model predictive control: A survey. *Robustness in Identification and Control*, 245:207–226, 1999.
2. Ch. S. Blagovest. *Hausdorff Approximations*. Springer, 1990.
3. M. Canale, L. Fagiano, and M.C. Signorile. On the robustness of receding horizon control using nonlinear approximated models. In *8th IFAC Symposium on Nonlinear Control Systems NOLCOS 2010*, Bologna, Italy, Septepmber 2010.

4. M. Canale and M. Milanese. Robust design of predictive controllers in presence of unmodeled dynamics. *European Journal of Control*, 9(5):499–506, 2003.
5. Qiu-Xia Chen, De-Feng He, and Li Yu. Input-to-state stability of min-max mpc scheme for nonlinear time-varying delay systems. *Asian Journal of Control*, 14(2):489501, 2012.
6. B. Ding. Properties of parameter-dependent open-loop mpc for uncertain systems with polytopic description. *Asian Journal of Control*, 12(1):58–70, 2010.
7. H. Genceli and M. Nikolaou. Robust stability analysis of constrained l_1-norm model predictive control. *AIChE Journal*, 39(12):1954–1956, 1993.
8. P. J. Goulart, E. C. Kerrigan, and J. M. Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42:523–533, 2006.
9. D. W. Jordan and P. Smith. *Nonlinear ordinary differential equations*. Oxford University Press, 1987.
10. M. Lazar, W.P.M.H. Heemels, B.J.P. Roset, H. Nijmeijer, and P.P.J. van den Bosch. Input-to-state stabilizing sub-optimal nmpc with an application to dc-dc converters. *International Journal of Robust and Nonlinear Control*, 18:890–904, 2008.
11. J.H. Lee and Z. Yu. Worst-case formulations of model predictive control for systems with bounded parameters. *Automatica*, 33(5):763–781, 1997.
12. C. Løvaas, M. M. Seron, and G. C. Goodwin. Robust output-feedback model predictive control for systems with unstructured uncertainty. *Automatica*, 44:1933–1943, 2008.
13. L. Magni, D.M. Raimondo, and R. Scattolini. Regional input-to-state stability for nonlinear model predictive control. *IEEE Transactions on automatic control*, 51(9):1548–1553, 2006.
14. D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.
15. M. Milanese and C. Novara. Set membership identification of nonlinear systems. *Automatica*, 40:957–975, 2004.
16. M. Milanese and C. Novara. Computation of local radius of information in SM–IBC identification of nonlinear systems. *Journal of Complexity*, 4(6):937–951, August 2007.
17. M. Morari and J.H. Lee. Model predictive control: past, present and future. *Computers and Chemical Engineering*, 23:667–682, 1999.
18. G. Pin, D.M. Raimondo, L. Magni, and T. Parisini. Robust model predictive control of nonlinear systems with bounded and state-dependent uncertainties. *IEEE Transactions on automatic control*, 57(7):1681–1687, 2009.
19. E. Zafiriou. Robust model predictive control of processes with hard constraints. In *Computers and Chemical Engineering*, volume 14, pages 359–371, 1990.
20. A. Zheng and M. Morari. Robust control of linear time-varying systems with constraints. In *Proceeding of the American Control Conference*, pages 2416–2420, 1994.