Efficient Train Operation via Shrinking Horizon Parametrized Predictive Control

Hafsa Farooqi *, **Lorenzo Fagiano** *, **Patrizio Colaneri** *,**

* *Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, Milano, Italy. E-mail addresses: hafsa.farooqi@polimi.it, lorenzo.fagiano@polimi.it, patrizio.colaneri@polimi.it*
** *IEIIT - CNR, Politecnico di Milano, Milano, Italy*

Abstract: The problem of driver assistance for the energy-efficient operation of trains is considered. The goal is to control the traction/braking forces applied to the train, while satisfying speed limits and reaching the next station at the prescribed arrival time. Moreover, the control input has to belong to a discrete set of values and/or operating modes, which a human driver has to implement. A nonlinear model predictive control (MPC) approach is proposed, featuring a shrinking horizon and an input-parametrization strategy to retain a continuous optimization problem. Theoretical convergence guarantees are derived, and the approach is tested in realistic simulations.

Keywords: Model Predictive Control, Input Parametrization, Nonlinear control systems, Train control

1. INTRODUCTION

Railway is by far the most efficient means of transportation from the point of view of energy consumption and therefore a strategic sector in today's society. The earliest existing works on efficient train control include developing strategies by considering the problem as a bounded state variable one (see Ichikawa (1968)) and development of numerical optimization techniques such as the ones presented by Milroy (1981) and Strobel and Horn (1973). In these approaches, linear expressions of resistance, constant constraints, non dependency of external forces on the train position, and constant slopes and track curvatures were considered. In reality, all these factors significantly affect the energy consumption. A method that considered variable slopes has been developed for underground trains with short station distance by Maksimov (1971). More recently, several techniques have been developed considering all these factors. A recent review of the existing techniques is provided by Scheepmaker et al. (2017).
For the energy efficient operation of railways, Model Predictive Control (MPC) is a suitable approach, thanks to its capability to deal with state and input constraints and economic objectives. MPC has been already applied in the context of this industrial application, see e.g. Aradi et al. (2014).

In this paper, we present a study on the application of a new MPC approach to the efficient operation of trains. In our research, carried out in collaboration with Alstom rail transport, the control input (usually the traction/braking force applied to the train) is constrained to belong to a set of discrete values or operating modes, which would naturally result in a mixed-integer nonlinear program to be solved at each sampling instant. To cope with this problem, the formulation we propose features an input parametrization strategy that yields a continuous optimization program. Moreover, we adopt a shrinking horizon, rather than the most common receding one. We named the resulting approach Shrinking horizon Parametrized Predictive Control (SPPC). We present both a nominal approach and two relaxed ones, where state constraints are softened to retain recursive feasibility. To the best of our knowledge, the combined presence of nonlinear dynamics, shrinking horizon, and input-parametrization strategy is new in the literature, and represents the main theoretical contribution of this paper, in addition to the application-specific content. Realistic simulation results show the effectiveness of the approach in the considered application.

2. PROBLEM STATEMENT AND ABSTRACTION

Consider an electric train controlled by a digital control unit. In this work, we consider space as the independent variable, while time will be one of the system's states. Thus, we denote with $k \in \mathbb{Z}$ the discrete space variable, and with D_s the sampling distance, so that the actual distance along the track at each sampling instant is equal to kD_s. This choice makes the control problem formulation easier in some respect. We denote with $x(k) = [x_1(k), x_2(k)]^T$ the state of the train, where x_1 is its travel time and x_2 the train speed ($\cdot^T$ denotes the matrix transpose operator), and with $u(k) \in [-1, 1]$ a normalized traction force, where $u(k) = 1$ corresponds to the maximum applicable traction and $u(k) = -1$ to the maximum braking. The input u is the available control variable. The train has to move from one station at time $x_1 = 0$ and reach the next one at time $x_1 = x_f$, covering the corresponding distance s_f. For a given pair of initial and final stations, the track features (slopes, curvature) are known in advance. Thus, in nominal conditions (i.e. with rated values of the train parameters, like its mass and the specifications of the powertrain and braking systems), according to Newton's laws and using the forward Euler discretization method, the equations of motion of a reasonably accurate model of this system read:

$$\begin{aligned} x_1(k+1) &= x_1(k) + \frac{D_s}{x_2(k)} \\ x_2(k+1) &= x_2(k) + D_s\left(\frac{F_T(x(k),u(k)) - F_B(x(k),u(k)) - F_R(k,x(k))}{Mx_2(k)}\right) \end{aligned} \tag{1}$$

where M is the total mass of the train, F_T is the traction force, F_B is the braking force, and F_R the resistive force. Functions $F_T(x,u)$, $F_B(x,u)$ are nonlinear and they depend on the specific train and track profile. They include, for example, look-up tables that link the traction and braking forces to the train speed and to the control input value. These functions are derived either experimentally or from complex models of the train and its traction and braking systems. In our research, these are provided by the business unit at Alstom. More details on these functions are omitted for confidentiality reasons. The resistive force $F_R(k,x)$ is also nonlinear, and it is the sum of a first term $R_v(x_2)$, accounting for resistance due to the velocity, and a second term $R_g(k)$, accounting for the effects of slopes and track curvature:

$$\begin{aligned} F_R(k,x) &= R_v(x_2) + R_g(k) \\ R_v(x_2) &= A + Bx_2 + Cx_2^2 \\ R_g(k) &= M_s\left(g\tan(\alpha(k)) + \frac{D}{r_c(k)}\right) \end{aligned} \tag{2}$$

where the parameters A,B,C,D are specific to the considered train, M_s is the static mass of the train, i.e. the mass calculated without taking into account the effective inertia of the rotating components, $r_c(k)$ and $\alpha(k)$ are, respectively, the track curvature and slope at position k, and g is the gravity acceleration. Besides the prescribed arrival time x_f and position s_f, there are additional state constraints that must be satisfied. These pertain to the limit on the maximum allowed velocity, $\overline{x}_2(k)$, which depends on the position k, since a different velocity limit is imposed for safety by the regulating authority according to the track features at each position. Overall, by defining the terminal space step as $k_f \doteq \lfloor s_f/D_s \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the flooring operation to the closest integer), the state constraints read:

$$\begin{array}{ll} x(0) & = [0,\, 0]^T \\ x(k_f) & = [x_f,\, 0]^T \\ x_2(k) & \geq 0,\; k = 0,\ldots,k_f \\ x_2(k) & \leq \overline{x}_2(k),\; k = 0,\ldots,k_f \end{array} \tag{3}$$

The control objective is to maximize the energy efficiency of the train while satisfying the constraints above. To translate this goal in mathematical terms, different possible cost functions can be considered. In our case, we consider the discretized integral of the absolute value of the traction power over space (with a constant scaling factor D_s^{-1}):

$$J = \sum_{k=0}^{k_f} |F_T(x(k),u(k))|\,. \tag{4}$$

This choice tends to produce controllers that minimize the traction energy injected into the system. The braking energy is not penalized, since in our case there is no restriction to the use of the braking system.

As already pointed out, the input variable is also constrained in the interval $u \in [-1,1]$. However, in a driver assistance scenario which is the main focus of this work, the control algorithm is developed to assist a human driver with a suggested value of the input handle, in which case only a smaller set of possible values can be delivered by the controller, in order to facilitate the human-machine interaction. In particular, in this scenario the input constraints are further tightened according to four possible operating modes prescribed by our industrial partner:

- *Acceleration:* in this mode, the input takes the maximum value, i.e. $u = 1$.
- *Coasting:* this mode implies that the traction is zero, i.e $u = 0$.
- *Cruising:* in this mode, the train engages a cruise control system that keeps a constant speed, i.e. u is computed by an inner control loop in such a way that $F_T = F_R$ for positive slopes and $F_B = F_R$ for negative slopes.
- *Braking:* in this mode the maximum braking force is used, i.e. $u = -1$.

Finally, a further feature of this application is a relatively small sampling space D_s with respect to the imposed overall space horizon s_f, resulting in a rather large number of sampling periodsin the interval $[0,\, s_f]$.

2.1 Problem Abstraction

The control problem described above can be cast in a rather standard form:

$$\min_{\boldsymbol{u}} \sum_{k=0}^{k_f} \ell(x(k),u(k)) \tag{5a}$$

subject to

$$x(k+1) = f(x(k),u(k)) \tag{5b}$$

$$u(k) \in U,\, k = 0,\ldots,k_f - 1 \tag{5c}$$

$$x(k) \in X,\, k = 1,\ldots,k_f \tag{5d}$$

$$x(0) = x_0 \tag{5e}$$

$$x(k_f) \in X_f \tag{5f}$$

where $x \in \mathbb{X} \subset \mathbb{R}^n$ is the system state, x_0 is the initial condition, $u \in \mathbb{U} \subset \mathbb{R}^m$ is the input, $f(x,u) : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is a known nonlinear mapping representing the system dynamics, and $l(x,u) : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is a stage cost function defined by the designer according to the control objective. The symbol $\boldsymbol{u} = \{u(0),\ldots,u(k_f-1)\} \in \mathbb{R}^{mk_f}$ represents the sequence of current and future control moves to be applied to the plant. The sets $X \subset \mathbb{X}$ and $U \subset \mathbb{U}$ represent the state and input constraints (including the discrete set of allowed inputs or input modes as described above), and the set $X_f \subset \mathbb{X}$ the terminal state constraints, which include a terminal equality constraint as a special case.

We recall that a continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathscr{K}$-function ($a \in \mathscr{K}$) if it is strictly increasing and $a(0) = 0$. Throughout this paper, we consider the following continuity assumption on the system model f.

Assumption 1. The function f enjoys the following continuity properties:

$$\begin{array}{l} \|f(x^1,u) - f(x^2,u)\| \leq a_x\left(\|x^1 - x^2\|\right),\, \forall x^1,x^2 \in \mathbb{X}, u \in \mathbb{U} \\ \|f(x,u^1) - f(x,u^2)\| \leq a_u\left(\|u^1 - u^2\|\right),\, \forall u^1,u^2 \in \mathbb{U}, x \in \mathbb{X} \end{array} \tag{6}$$

where $a_x,\, a_u \in \mathscr{K}$. □

In (6) and in the remainder of this paper, any vector norm $\|\cdot\|$ can be considered. Assumption (1) is reasonable in most real-world applications, and it holds in the railway application considered here.

3. NOMINAL SPPC APPROACH

To solve problem (5), we resort to Nonlinear Model Predictive Control (NMPC) with shrinking horizon and input-parametrization strategy to reduce the computational burden required by the feedback controller as well as to enforce the discrete constraints on the input while retaining a continuous optimization program, instead of the mixed-integer one that would result from direct optimization of the input.

3.1 Parametrization Setup

We adopt a parametrization of the control input that allows us to naturally incorporate the presence of the discrete (switched) driving modes described in Section 2. The main idea is to first split the track (i.e. the whole prediction horizon from 0 to k_f) into sectors. Then, in each sector, we pre-define a switching sequence of the four driving modes, and we optimize over the switching (space) instants of the sequence. In this way, we retain a continuous vector of optimization variables, thus improving the computational efficiency, while still providing the controller with enough degrees-of-freedom to optimize the predicted system behavior. Specifically, we consider a number $s_n \in \mathbb{N}$ of sectors, each one with length Γ_i, such that

$$\sum_{i=1}^{s_n} \Gamma_i = s_f. \tag{7}$$

The choice of sectors is carried out by considering characteristics such as the presence of constant velocity limits and the resistance force due to the slopes and track curvature $R_g(k)$, which are known in advance, see Fig. 1 for an example. Regarding the switching sequence ("driving style") adopted in each sector, denoted with $\boldsymbol{u}_{d_i}$, we choose the following:

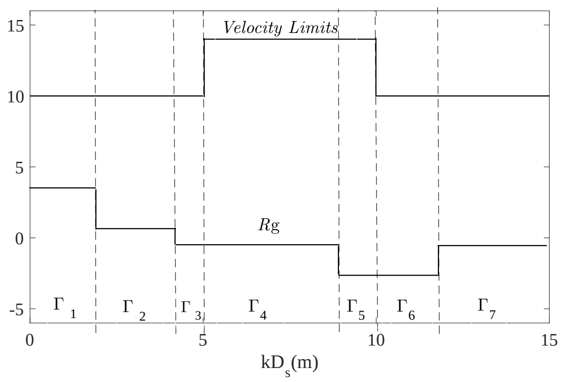

Fig. 1. Example of sector choice for a track with $s_f = 15$ and s_n= 7. Possible sectors based on similar characteristics such as velocity limits and R_g values are depicted.

$$\begin{aligned} \boldsymbol{u}_{d_i} &= \{\ u_{(i,1)},\ u_{(i,2)},\ u_{(i,3)},\ u_{(i,4)}\ \} \\ &= \{\quad 1,\quad u_{CR},\quad 0,\quad -1\ \},\ \forall i \in [1,\dots,s_n] \end{aligned} \tag{8}$$

where $u_{(i,\ell)}$ is the input issued in the $\ell-$th phase of the $i-$th sector (with $\ell \in \{1,2,3,4\}$) and u_{CR} identifies the cruising mode (see Section 2), i.e. where the actual input $u(k)$ is computed by a cruise control system in order to maintain a constant speed. The chosen switching sequence is such that in each sector, theoretically, the controller can choose to have a traction phase, a cruising one, a coasting one, and finally a braking phase. Thus, the maximum number of operating modes that can be set in the whole optimization horizon is equal to $4s_n$. As anticipated, our optimization variables will be the switching instants, or more precisely the duration of each phase within each switching sequence. We denote these quantities with $\delta s_{(i,\ell)}$, such that:

$$\begin{aligned} &0 \le \delta s_{(i,\ell)} \le \Gamma_i \\ &\sum_{\ell=1}^{4} \delta s_{(i,\ell)} = \Gamma_i. \end{aligned} \tag{9}$$

As an example, values of $(s_{(i,1)}, s_{(i,2)}, s_{(i,3)}, s_{(i,4)})$ equal to $(0,0,\Gamma_i,0)$ correspond to the train coasting throughout sector i, and so on.

Remark 1. The presented parametrization provides the controller with enough degrees of freedom to choose a single mode or multiple modes of operation in each sector, in order to compensate the presence of uncertainty and to adapt to the track features and constraints. Additionally, the constraint on prescribed arrival position needs to be satisfied: this is done implicitly by imposing (9) for each sector, which implies (7).

We are now in position to introduce the optimal control problem to be solved at each time step in our shrinking horizon approach. We named the resulting control approach Shrinking horizon Parametrized Predictive Control (SPPC), presented next.

3.2 Shrinking horizon Parametrized Predictive Control (SPPC)

We start by defining the optimization variables available at each step k. The set of indexes identifying the current and future sectors, from the current position kD_s until the end of the track s_f, is given by:

$$\{i : \underline{i}(k) + 1 \le i \le s_n\}, \tag{10}$$

where

$$\underline{i}(k) \doteq \begin{cases} \max\limits_{\bar{i} \ge 1} \bar{i} \quad \text{s.t.} \ \sum\limits_{i=1}^{\bar{i}} \Gamma_i < kD_s, & \text{if} \ \ \Gamma_1 < kD_s \\ 0, & \text{otherwise} \end{cases}$$

Then, the number $N(k)$ of free variables to be computed corresponds to the number of remaining sectors, equal to $(s_n - \underline{i}(k))$, times the number of modes in each sector, i.e. 4 in our case. Therefore, we have $N(k) = 4(s_n - \underline{i}(k))$. We denote the vector of optimization variables with

$$\boldsymbol{v}_{N(k)} \doteq \{\delta s_{(\underline{i}(k)+1,1)}, \dots, \delta s_{(\underline{i}(k)+1,4)}, \dots, \delta s_{(s_n,4)}\}^T \in \mathbb{R}^{N(k)}. \tag{11}$$

Let us indicate with $u(j|k)$ the input vector at each space sample $k+j$ predicted at step k. Considering the parametrization described in Section 3.1, we can define the function $g(\boldsymbol{v}_{N(k)}, j, k)$ that links, at each step k and for each $j = 0, \dots, k_f - k - 1$, the optimization variables $\boldsymbol{v}_{N(k)}$ with the predicted input $u(j|k)$:

$$u(j|k) = g(\boldsymbol{v}_{N(k)}, j, k) \doteq u_{(\hat{i}(j,k), \hat{\ell}(\boldsymbol{v}_{N(k)}, j, k))}, \tag{12}$$

where (compare with (8)):

$$\begin{aligned} \hat{i}(j,k) \doteq &\min_{\bar{i}=1,\dots,s_n} \ \bar{i} \\ &\text{s.t.} \\ &\sum_{i=1}^{\bar{i}} \Gamma_i \ge (k+j)D_s \end{aligned}$$

and

$$\begin{aligned} \hat{\ell}(\boldsymbol{v}_{N(k)}, j, k) \doteq &\min_{\bar{\ell}=1,\dots,4} \ \bar{\ell} \\ &\text{s.t.} \\ &\sum_{i=1}^{\hat{i}(j,k)} \Gamma_i + \sum_{\ell=1}^{\bar{\ell}} \delta s_{(\hat{i}(j,k),\ell)} \ge (k+j)D_s. \end{aligned}$$

Note that the evaluation of (12) is very efficient, since it just amounts to finding, for each space sample $k+j$, the indexes of the corresponding sector and phase and then to apply the corresponding pre-defined driving mode from (8). Finally, let us denote with $x(j|k)$, $j = 0, ..., k_f - k$ the state vectors predicted at space sample $k+j$ starting from the one at step k. At each step $k \in [0, k_f - 1]$, we formulate the following FHOCP:

$$\min_{\boldsymbol{v}_{N(k)}} \sum_{j=0}^{k_f-k} \ell(x(j|k), u(j|k)) \tag{13a}$$
subject to
$$u(j|k) = g(\boldsymbol{v}, \boldsymbol{v}_{N(k)}, j, k),\ j = 0, \ldots, k_f - k - 1 \tag{13b}$$
$$x(j+1|k) = f(x(j|k), u(j|k)),\ j = 0, \ldots, k_f - k - 1 \tag{13c}$$
$$u(j|k) \in U,\ j = 0, \ldots, k_f - k - 1 \tag{13d}$$
$$x(j|k) \in X,\ j = 1, \ldots, k_f - k \tag{13e}$$
$$x(0|k) = x(k) \tag{13f}$$
$$A(k)\boldsymbol{v}_{N(k)} \leq b(k) \tag{13g}$$
$$x(k_f - k|k) \in X_f \tag{13h}$$

Where the matrices $A(k), b(k)$ in (13g) are built to enforce the constraints (9). We denote with $\boldsymbol{v}^*_{N(k)} = \{\delta s^*_{(\underline{i}(k)+1,1)}, \ldots, \delta s^*_{(s_n,4)}\}^T$ a solution (in general only locally optimal) of (13). Moreover, we denote with $\boldsymbol{x}^*(k)$ and $\boldsymbol{u}^*(k)$ the corresponding predicted sequences of state and input vectors:

$$\boldsymbol{x}^*(k) = \{x^*(0|k), \ldots, x^*(k_f - k|k)\} \tag{14a}$$
$$\boldsymbol{u}^*(k) = \{u^*(0|k), \ldots, u^*(k_f - 1 - k|k)\} \tag{14b}$$
where
$$x^*(0|k) = x(k)$$
$$x^*(j+1|k) = f(x^*(j|k), u^*(j|k)) \tag{14c}$$
$$u^*(j|k) = g(\boldsymbol{v}^*_{N(k)}, j, k) \tag{14d}$$

The SPPC strategy is obtained by recursively solving (13), as described by the following pseudo-algorithm.

Algorithm 1. Nominal SPPC strategy

(1) At sampling instant k, measure or estimate the state $x(k)$ and solve the FHOCP (13). Let $\boldsymbol{v}^*_{N(k)}$ be the computed solution;
(2) Apply to the plant the first element of the control sequence $\boldsymbol{u}^*(k)$, i.e. $u(k) = u^*(0|k) = g(\boldsymbol{v}^*_{N(k)}, 0, k)$;
(3) Repeat the procedure from 1) at the next sampling period. □

Algorithm 1 defines the following feedback control law:

$$u(k) = \mu(x(k)) = u^*(0|k), \tag{15}$$

and the resulting model of the closed-loop system is:

$$x(k+1) = f(x(k), \mu(x(k))) \tag{16}$$

The recursive feasibility of (13) and convergence of the state of (16) to the terminal set is established by construction of the nominal SPPC strategy. In the next section, we introduce a model of uncertainty, whose form is motivated again by the application described in Section 2, and a possible variation of Algorithm 1 to deal with it, along with its guaranteed convergence properties. We term this variation the "relaxed" approach, since it involves the use of suitable soft (i.e. relaxed) constraints to guarantee recursive feasibility.

4. RELAXED SPPC APPROACH: ALGORITHM AND PROPERTIES

To model uncertainty and disturbances we consider an additive term $d(k)$ acting on the input vector, i.e.:

$$\tilde{u}(k) = u(k) + d(k) \tag{17}$$

where $\tilde{u}(k)$ is the disturbance-corrupted input provided to the plant. This model represents well all cases where plant uncertainty and exogenous disturbances can be translated into an effect similar to the control input (the so-called matched uncertainty). For example, in our application, equation (17) can describe uncertainty in the train mass, drivetrain specs, track slope and misapplication of the computed input by the human operator in a driver assistance scenario (see Section 2).
We consider the following assumption on d:

Assumption 2. The disturbance term d belongs to a compact set $\mathbb{D} \subset \mathbb{R}^m$ such that:

$$\|d\| \leq \overline{d},\ \forall d \in \mathbb{D} \tag{18}$$

where $\overline{d} \in (0, +\infty)$. □

This assumption holds in many practical cases and in the considered train application as well. We indicate the perturbed state trajectory due to the presence of d as:

$$\tilde{x}(k+1) = f(\tilde{x}(k), \tilde{u}(k)),\ k = 0, \ldots, k_f$$

where $\tilde{x}(0) = x(0)$. To retain recursive feasibility in presence of the disturbance, we soften the constraints in the FHOCP. Since, in our railway application the terminal state constraint is the most important one from the viewpoint of system performance, to be more specific we restrict our analysis to the terminal state constraint in (5f), i.e. $x(k_f) \in X_f$ without loss of generality. The other constraints (velocity limits) are always enforced for safety by modulating traction or by braking.

Let us denote the distance between a point x and a set X as:

$$\Delta(x, X) = \min_{y \in X} \|x - y\|.$$

Then, we want to derive a modified SPPC strategy with softened terminal state constraint (to ensure recursive feasibility) that guarantees a property of the following form in closed loop:

$$\Delta(\tilde{x}(k_f), X_f) \leq \beta(\overline{d}),\ \beta \in \mathscr{K}. \tag{19}$$

That is, the distance between the terminal state and the terminal constraint is bounded by a value that decreases strictly to zero as $\overline{d} \to 0$. In order to obtain this property, we propose a relaxed SPPC approach using a two-step constraint softening procedure, described next.

4.1 Two-step relaxed SPPC strategy

At each step k we consider a strategy consisting of two optimization problems to be solved in sequence:

a) we compute the best (i.e. smallest) achievable distance between the terminal state and the terminal set, starting from the current perturbed state $\tilde{x}(k)$:

$$\underline{\gamma} = \arg \min_{\boldsymbol{v}_{N(k)},\gamma} \gamma$$
$$\text{subject to}$$
$$u(j|k) = g(\boldsymbol{v}_{N(k)}, j, k),\ j = 0,\ldots,k_f - k - 1$$
$$x(j+1|k) = f(x(j|k), u(j|k)),\ j = 0,\ldots,k_f - k - 1$$
$$u(j|k) \in U,\ j = 0,\ldots,k_f - k - 1$$
$$x(j|k) \in X,\ j = 1,\ldots,k_f - k$$
$$x(0|k) = \tilde{x}(k)$$
$$A(k)\boldsymbol{v}_{N(k)} \leq b(k)$$
$$\Delta(x(k_f - k|k), X_f) \leq \gamma \tag{20}$$

b) we optimize the input sequence using the original cost function, and softening the terminal constraint by $\underline{\gamma}$:

$$\min_{\boldsymbol{v}_{N(k)}} \sum_{j=0}^{k_f-k} \ell(\tilde{x}(j|k), u(j|k))$$
$$\text{subject to}$$
$$u(j|k) = g(\boldsymbol{v}_{N(k)}, j, k),\ j = 0,\ldots,k_f - k - 1$$
$$x(j+1|k) = f(x(j|k), u(j|k)),\ j = 0,\ldots,k_f - k - 1$$
$$u(j|k) \in U,\ j = 0,\ldots,k_f - k - 1$$
$$x(j|k) \in X,\ j = 1,\ldots,k_f - k$$
$$x(0|k) = \tilde{x}(k)$$
$$A(k)\boldsymbol{v}_{N(k)} \leq b(k)$$
$$\Delta(x(k_f - k|k), X_f) \leq \underline{\gamma} \tag{21}$$

By construction, both problems are always feasible (with the caveat that state constraints are considered to be always feasible, as discussed above, otherwise the softening shall be applied to these constraints as well). We denote with $\boldsymbol{v}^r_{N(k)}$, $\boldsymbol{x}^r(k)$ and $\boldsymbol{u}^r(k)$ the optimized sequences of decision variables, state and inputs resulting from the solution of (21). The sequences $\boldsymbol{x}^r(k)$ and $\boldsymbol{u}^r(k)$ are computed from $\boldsymbol{v}^r_{N(k)}$ and $\tilde{x}(k)$ as reported in (14). The resulting relaxed SPPC strategy is implemented by the following pseudo-algorithm.

Algorithm 2. Two-stage relaxed SPPC strategy

(1) At sampling instant k, measure or estimate the state $\tilde{x}(k)$ and solve in sequence the optimization problems (20)-(21). Let $\boldsymbol{v}^r_{N(k)}$ be the computed solution;
(2) Apply to the plant the control vector $u(k) = u^r(0|k) = g(\boldsymbol{v}^r_{N(k)}, 0, k)$;
(3) Repeat the procedure from (1) at the next sampling period. □

Algorithm 2 defines the following feedback control law:

$$u(k) = \mu^r(\tilde{x}(k) = u^r(0|k), \tag{22}$$

and the resulting closed-loop dynamics are given by:

$$\tilde{x}(k+1) = f(\tilde{x}(k), \mu^r(\tilde{x}(k)) + d(k)). \tag{23}$$

The next result shows that the closed-loop system (23) enjoys a uniformly bounded accuracy property of the form (19), provided that the nominal SPPC problem (13) is feasible at $k = 0$.

Theorem 1. Let Assumptions 1 and 2 hold and let the FHOCP (13) be feasible at step $k = 0$. Then, the terminal state $\tilde{x}(k_f)$ of system (23) enjoys property (19) with

$$\Delta(\tilde{x}(k_f), X_f) \leq \beta(\overline{d}) = \sum_{k=0}^{k_f-1} \beta_{k_f-k-1}(\overline{d}) \tag{24}$$

where

$$\begin{aligned} \beta_0(\overline{d}) &= a_u(\overline{d}) \\ \beta_k(\overline{d}) &= a_u(\overline{d}) + a_x(\beta_{k-1}(\overline{d})),\ k = 1,\ldots,k_f - 1 \end{aligned} \tag{25}$$

■

The proof is by induction and is very similar to the proof of Theorem 1 in Farooqi et al. (2018) and hence has been removed for brevity. Theorem 1 indicates that the worst-case distance between the terminal state and the terminal set is bounded by a value which is zero for $\overline{d} = 0$ and increases strictly with the disturbance bound. In the considered railway application this means that, for example, the worst-case accuracy degradation in reaching the terminal station on time due to the changing mass of the train which is a function of the passenger load, is proportional to the largest difference between the nominal mass and its change due to the varying passenger load. This result provides a theoretical justification to the proposed two-step relaxed SPPC approach. The bound (24) is conservative, since it essentially results from the accumulation of worst-case perturbations induced by the disturbance on the open-loop trajectories computed at each k. As we show in our simulation results, in practice the resulting closed-loop performance are usually very close to those of the nominal case, thanks to recursive optimization in the feedback control loop.

4.2 *Multi-objective relaxed SPPC strategy*

As an alternative to the two-step approach described above, one can also consider a multi-objective minimization:

$$\min_{\boldsymbol{v}_{N(k)},\beta} \sum_{j=0}^{k_f-k} \ell(\tilde{x}(j|k), u(j|k)) + \omega\gamma$$
$$\text{subject to}$$
$$u(j|k) = g(\boldsymbol{v}_{N(k)}, j, k),\ j = 0,\ldots,k_f - k - 1$$
$$x(j+1|k) = f(\tilde{x}(j|k), u(j|k)),\ j = 0,\ldots,k_f - k - 1 \tag{26}$$
$$u(j|k) \in U,\ j = 0,\ldots,k_f - k - 1$$
$$x(j|k) \in X,\ j = 1,\ldots,k_f - k$$
$$x(0|k) = \tilde{x}(k)$$
$$A(k)\boldsymbol{v}_{N(k)} \leq b(k)$$
$$\Delta(x(k_f - k|k), X_f) \leq \gamma$$

where ω is a positive weight on the scalar γ. Problem (26) can be solved in Algorithm (2) in place of problems (20)-(21). In this case, the advantage is that a trade-off between constraint relaxation and performance can be set by tuning ω. Regarding the guaranteed bounds on constraint violation, with arguments similar to those employed in Fagiano and Teel (2013) one can show that, at each $k \in [0, k_f - 1]$, for any $\varepsilon > 0$ there exists a finite value of ω such that the distance between the terminal state and the terminal set is smaller than $\gamma_{k_f-k-1}(\overline{d}) + \varepsilon$. Thus, with large-enough ω, one can recover the behavior obtained with the two-step relaxed SPPC approach. The theoretical derivation is omitted for the sake of brevity, as it is a rather minor extension of the results of Fagiano and Teel (2013).

5. SIMULATION RESULTS

We tested the proposed strategies in realistic simulations with real train data provided by Alstom for a section of the Amsterdam metro rail, in particular the track between Rokin and Central Station. The parametric values of the train used in the controller are (see (1)-(2)) $M = 142403\,\text{kg}$, $M_s = 131403\,\text{kg}$, $A = 3975.9\,\text{N}$, $B = 24.36\,\text{Nsm}^{-1}$ and $C = 4.38\,\text{Nsm}^{-2}$, while the maximum traction and braking forces allowed for this particular train are in the form of look up tables (see Fig. 2). These forces are of the form $F_T(x(k), u(k)) = F_{Tmax}(x_2(k))u(k)$, and $F_B(x(k),u(k)) = F_{Bmax}(x_2(k))u(k)$. The input variable is constrained in the set $[-1, 1]$.

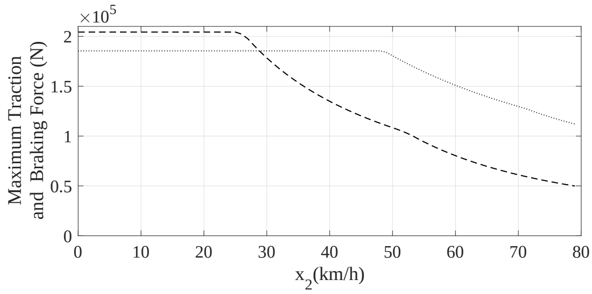

Fig. 2. Maximum traction force F_{Tmax}(dashed) and braking force F_{Bmax} (dotted) allowed for the train considered in the simulation.

The considered track has zero curvature, slopes as plotted in Fig. 3, and velocity limits reported in Fig. 4.

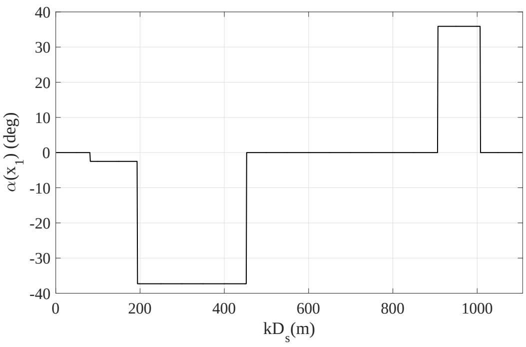

Fig. 3. Slopes of the Amsterdam metro track segment considered in the simulation.

The train has to reach the next station at $s_f = 1106\,\text{m}$ in $x_f = 76\,\text{s}$. The track is divided into $s_n = 15$ sectors. The employed sampling distance is $D_s = 18.4\,\text{m}$.

We compare a nominal situation, where Algorithm 1 is applied and no model uncertainty is present, with a case where we employ Algorithm 2 (with both the two-stage approach and the multi-objective optimization variants) in presence of random parameter uncertainty ($\pm 10\%$ of each model parameter). For the multi-objective approach, we set $\omega = 160$ (see (26)).

The obtained velocity profiles are presented in Fig. 4. From the plots, it is evident that in order to save energy, after accelerating to a certain velocity, the train mostly coasts or cruises taking advantage of the slopes. All the velocity constraints are always satisfied.

Regarding the obtained final time between the terminal state and the target one at $k = k_f$, in the presence of uncertainty this is delayed by $2\,\text{s}$ (i.e. $x_f = 78\,\text{s}$) for both the 2-stage approach and the multi-objective one. This result is perfectly compatible with the desired performance in this application. Fig. 4 presents also the "all-out" solution, which achieves the shortest arrival time compatible with all the constraints. This corresponds to $x_f = 74.2\,\text{s}$.

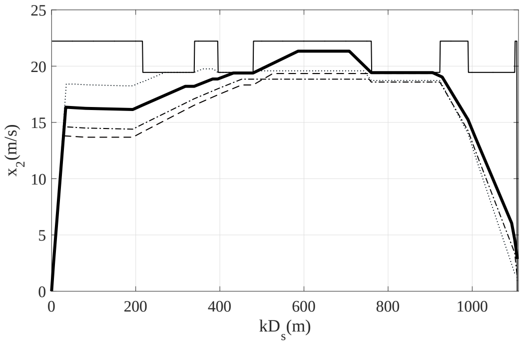

Fig. 4. Simulation results. Velocity profiles as a function of the train position obtained with different SPPC strategies: nominal (dotted line), two-step relaxed (dashed), multi-objective (dash-dot). The "all-out" solution ($x_f = 74.2\,\text{s}$) is shown with a thick solid line, and velocity limits with a thin solid line.

6. CONCLUSIONS

We considered the problem of energy-efficient train operation, involving a finite terminal position and corresponding state constraint, and a discrete set of allowed input values or operating modes. To address this problem, we proposed a MPC approach with a shrinking horizon and a particular input parametrization, which allows one retain a continuous optimization problem at each step. We derived convergence guarantees in both nominal conditions and under uncertainty, and showcased the technique in realistic simulations.

ACKNOWLEDGEMENTS

The authors thank Alstom rail transport for providing the model and track data employed in the simulation study.

REFERENCES

Aradi, S., Bécsi, T., and Gáspár, P. (2014). Design of predictive optimization method for energy-efficient operation of trains. In *European Control Conference (ECC)*, 2490–2495.

Fagiano, L. and Teel, A. (2013). Generalized terminal state constraint for model predictive control. *Automatica*, 49(5), 2622–2631.

Farooqi, H., Fagiano, L., and Colaneri, P. (2018). On shrinking horizon move-blocking predictive control. *Submitted to Automatica, Preliminary version available on arXiv*, identifier/1803.09676. URL `http://arxiv.org/abs/1803.09676`.

Ichikawa, K. (1968). Application of optimization theory for bounded state variable problems to the operation of train. *Bulletin of JSME*, 11(47), 857–865.

Maksimov, V. (1971). Optimal control of automatic subway trains. *Proceedings of Moscow Railway Engineering Institute (Trudy MIIT)*, (388), 103–114.

Milroy, I.P. (1981). *Minimum-energy control of rail vehicles*. South Australian Institute of Technology.

Scheepmaker, G.M., Goverde, R.M., and Kroon, L.G. (2017). Review of energy-efficient train control and timetabling. *European Journal of Operational Research*, 257(2), 355–376.

Strobel, H. and Horn, P. (1973). On energy-optimum control of train movement with phase constraints. *Electric, Informatics and Energy Technique Journal*, 6, 304–308.