The Scenario Approach for Stochastic Model Predictive Control with Bounds on Closed-Loop Constraint Violations ⋆

Georg Schildbach [a], Lorenzo Fagiano [a,b], Christoph Frei [c], Manfred Morari [a]

[a] *Automatic Control Laboratory, Swiss Federal Institute of Technology Zurich (Physikstrasse 3, 8092 Zurich, Switzerland)*

[b] *ABB Switzerland Ltd., Corporate Research (Segelhofstrasse 1, Baden-Daettwil, Switzerland)*

[c] *Mathematical and Statistical Sciences, University of Alberta (Edmonton, AB T6G 2G1, Canada)*

Abstract

Many practical applications in control require that constraints on the inputs and states of the system are respected, while some performance criterion is optimized. In the presence of model uncertainties or disturbances, it is often sufficient to satisfy the state constraints for at least a prescribed share of the time, such as in building climate control or load mitigation for wind turbines. For such systems, this paper presents a new method of Scenario-Based Model Predictive Control (SCMPC). The basic idea is to optimize the control inputs over a finite horizon, subject to robust constraint satisfaction under a finite number of random scenarios of the uncertainty and/or disturbances. Previous SCMPC approaches have suffered from a substantial gap between the rate of constraint violations specified in the optimal control problem and that actually observed in closed-loop operation of the controlled system. This paper identifies the two theoretical explanations for this gap. First, accounting for the special structure of the optimal control problem leads to a substantial reduction of the problem dimension. Second, the probabilistic constraints have to be interpreted as average-in-time, rather than pointwise-in-time. Based on these insights, a novel SCMPC method can be devised for general linear systems with additive and multiplicative disturbances, for which the number of scenarios is significantly reduced. The presented method retains the essential advantages of the general SCMPC approach, namely a low computational complexity and the ability to handle arbitrary probability distributions. Moreover, the computational complexity can be adjusted by a sample-and-remove strategy.

Key words: Scenario-Based MPC; Stochastic MPC; Soft Constraints; Stochastic Systems; Scenario Optimization; Chance Constraints

1 Introduction

Model Predictive Control (MPC) is a powerful approach for handling multi-variable control problems with constraints on the states and inputs. Its feedback control law can also incorporate feedforward information, e.g. about the future course of references and/or disturbances, and the optimization of a performance criterion of interest.

Over the past two decades, the theory of linear and robust MPC has matured considerably [22]. There are also widespread practical applications in diverse fields [26]. Yet many potentials of MPC are still not fully uncovered.

One active line of research is Stochastic MPC (SMPC), where the system dynamics are of a stochastic nature. They may be affected by additive disturbances [3, 10, 13, 14, 18, 19], by random uncertainty in the system matrices [11], or both [12, 15, 25, 30]. The common objective is to design a controller that minimizes a given stage cost criterion for the closed-loop system. Moreover, the controller must choose its actions from a constrained set of inputs and observe *chance constraints* for the system state (i.e., constraints that have to be satisfied only with a given probability).

Stochastic systems with chance constraints arise naturally in some applications, such as building climate control [23], wind turbine control [12], or network traffic

⋆ This paper was not presented at any IFAC meeting. Corresponding author Georg Schildbach. Tel. +41-4463-24279. Fax +41-4463-21211.

Email addresses: schildbach@control.ee.ethz.ch (Georg Schildbach), lorenzo.fagiano@ch.abb.com (Lorenzo Fagiano), cfrei@ualberta.ca (Christoph Frei), morari@control.ee.ethz.ch (Manfred Morari).

control [34]. Alternatively, they can be considered as relaxations of robust control problems, in which the robust satisfaction of state constraints can be traded for an improved performance (in terms of the cost function).

A major challenge in many SMPC approaches is the solution to chance-constrained finite-horizon optimal control problems (FHOCPs) in each sample time step. These correspond to non-convex stochastic programs, for which finding an exact solution is computationally intractable, except for very special cases [17, 31]. Moreover, due to the multi-stage nature of these problems, they generally involve the computation of multi-variate convolution integrals [10].

In order to obtain a tractable solution, various sample-based approximation approaches have been considered, e.g. [2, 4, 32]. They share the significant advantage of coping with generic probability distributions, as long as a sufficient number of random samples (or '*scenarios*') can be obtained. The open-loop control laws can be approximated by sums of basis functions, as in the Q-design procedure proposed by [32]. However, these early approaches of Scenario-Based MPC (SCMPC) remain computationally demanding [2] and/or of a heuristic nature; that is, without specific guarantees on the satisfaction of the chance constraints [4, 32].

More recent SCMPC approaches [6, 7, 21, 24, 28, 33] are based on theoretical advances in the field of scenario-based optimization [8, 9]. Some of these approaches focus on obtaining a relaxation of a Robust MPC solution [6, 7, 33]. Others aim for the satisfaction of chance constraints on the system state in the Stochastic MPC sense [21, 24, 28]. For the latter approaches, a major difficulty lies in identifying the right sample size corresponding to a prescribed level of constraint violations. In particular, all theoretical bounds used in the past have been found to be (much) higher than the sample sizes that were identified by empirical tests. This effect has been referred to as '*conservatism*' in some of the literature [7, 21].

The contribution of this work is to reduce the theoretically required sample size by eliminating the two main causes for this conservatism. First, a new framework is introduced in which the chance constraints are interpreted as a time average, rather than pointwise-in-time with a high confidence. Second, the novel approach exploits the structural properties of the finite-horizon optimal control problem [29], which also allows for the presence of multiple simultaneous chance constraints on the state. As some of the previous SCMPC approaches [21, 24], the presented method may be augmented by a scheme for a-posteriori removal of adverse samples. Such a scheme tends to improve the performance of the controller, while it increases the computational cost.

The paper is organized as follows. Section 2 presents a rigorous formulation of the optimal control problem that one would like to solve. Section 3 describes how an approximated solution is obtained by SCMPC. Section 4 develops the theoretical details, including the technical background and closed-loop properties. Section 5 demonstrates the application of the method to a numerical example. Finally, Section 6 presents the main conclusions.

2 Optimal Control Problem

Consider a discrete-time control system with a linear stochastic transition map

$$x_{t+1} = A(\delta_t)x_t + B(\delta_t)u_t + w(\delta_t) \ , \quad x_0 = \bar{x}_0 \ , \quad (1)$$

for some fixed initial condition $\bar{x}_0 \in \mathbb{R}^n$. The *system matrix* $A(\delta_t) \in \mathbb{R}^{n\times n}$ and the *input matrix* $B(\delta_t) \in \mathbb{R}^{n\times m}$ as well as the additive disturbance $w(\delta_t) \in \mathbb{R}^n$ are random, as they are (known) functions of a primal uncertainty δ_t. For notational simplicity, δ_t comprises all uncertain influences on the system at time $t \in \mathbb{N}$.

Assumption 1 (Uncertainty) *(a) The uncertainties* $\{\delta_0, \delta_1, ...\}$*, are independent and identically distributed (i.i.d.) random variables on a probability space* $(\Delta, \mathbf{P})$*. (b) A 'sufficient number' of i.i.d. samples from* δ_t *can be obtained, either empirically or by a random number generator.*

The support set Δ of δ_t and the probability measure $\mathbf{P}$ on Δ are entirely generic. In fact, Δ and $\mathbf{P}$ need not be known explicitly. The 'sufficient number' of samples, which is required instead, will become concrete in later sections of the paper. Note that any issues arising from the definition of a σ-algebra on $(\Delta, \mathbf{P})$ are glossed over in this paper, as they are unnecessarily technical. Instead, every relevant subset of Δ is assumed to be measurable.

The system (1) can be controlled by inputs $\{u_0, u_1, ...\}$, to be chosen from a set of feasible inputs $\mathbb{U} \subset \mathbb{R}^m$. Since the future evolution of the system (1) is uncertain, it is generally impractical to indicate all future inputs explicitly. Instead, each u_t should be determined by a static feedback law

$$: \mathbb{R}^n \to \mathbb{U} \qquad \text{with} \qquad u_t = \psi(x_t) \ ,$$

based only on the current state of the system.

The optimal state feedback law $\psi(\cdot)$ should be determined in order to minimize the time-average of expected stage costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{0+}$,

$$\frac{1}{T}\sum_{t=0}^{T-1} \mathbf{E}\big[\ell\big(x_t, u_t\big)\big] \ . \quad (2)$$

Each stage cost is taken in expectation $\mathbf{E}[\,\cdot\,]$, since its arguments x_t and u_t are random variables, being functions of $\{\delta_0, ..., \delta_{t-1}\}$. The time horizon T is considered to be very large, yet it may not be precisely known at the point of the controller design.

The minimization of the cost is subject to keeping the state inside a state constraint set $\mathbb{X}$ for a given fraction of all time steps. For many applications, the robust satisfaction of the state constraint (i.e. $x_t \in \mathbb{X}$ at all times t) is too restrictive for the choice of $\psi(\cdot)$, and results in a poor performance in terms of the cost function. This is especially true in cases where the lowest values of the cost function are achieved close to the boundary of $\mathbb{X}$. Moreover, robust satisfaction of the state constraint may be impossible to enforce if the support of $w(\delta_t)$ is unknown and possibly unbounded.

In order to make this more precise, let $M_t := \mathbf{1}_{\mathbb{X}^{\mathrm{C}}}(x_{t+1})$ denote the random variable indicating that $x_{t+1} \notin \mathbb{X}$. Here $\mathbf{1}_{\mathbb{X}^{\mathrm{C}}} : \mathbb{R}^n \to \{0, 1\}$ is the indicator function on the complement $\mathbb{X}^{\mathrm{C}}$ of $\mathbb{X}$. The goal is for the expected time-average of constraint violations to be upper bounded by a given *violation level* $\varepsilon \in (0, 1)$,

$$\mathbf{E}\Big[\frac{1}{T}\sum_{t=0}^{T-1} M_t\Big] \leq \varepsilon \ . \tag{3}$$

Assumption 2 (Control Problem) *(a) The state of the system can be measured at each time step t. (b) The set of* feasible inputs $\mathbb{U}$ *is bounded and convex. (c) The* state constrained set $\mathbb{X}$ *is convex. (d) The stage cost* $\ell(\cdot,\cdot)$ *is a convex function.*

Assumption 2(b) holds for most practical applications, and very large artificial bounds can always be introduced for input channels without natural bounds. Typical choices for the stage cost ℓ include

$$\ell(\xi, \upsilon) := \left\|Q_\ell \xi\right\|_1 + \left\|R_\ell \upsilon\right\|_1 \ , \tag{4a}$$

$$\text{or} \quad \ell(\xi, \upsilon) := \left\|Q_\ell \xi\right\|_\infty + \left\|R_\ell \upsilon\right\|_\infty \ , \tag{4b}$$

$$\text{or} \quad \ell(\xi, \upsilon) := \left\|Q_\ell \xi\right\|_2^2 + \left\|R_\ell \upsilon\right\|_2^2 \ , \tag{4c}$$

where $Q_\ell \in \mathbb{R}^{n\times n}$ and $R_\ell \in \mathbb{R}^{m\times m}$ are positive semi-definite weighting matrices. Typical choices for the constraints $\mathbb{U}$ and $\mathbb{X}$ are polytopic or ellipsoidal sets.

Combining the previous discussions, the *optimal control problem (OCP)* can be stated as follows:

$$\min_{\psi(\cdot)} \quad \frac{1}{T}\sum_{t=0}^{T-1} \mathbf{E}\big[\ell\big(x_t, u_t\big)\big] \ , \tag{5a}$$

$$\text{s.t.} \ \ x_{t+1} = A(\delta_t)x_t + B(\delta_t)u_t + w(\delta_t) \ , \ \ x_0 = \bar{x}_0 \quad \forall\, t = 0, ..., T-1 \ , \tag{5b}$$

$$\mathbf{E}\Big[\frac{1}{T}\sum_{t=0}^{T-1} \mathbf{1}_{\mathbb{X}^{\mathrm{C}}}(x_t)\Big] \leq \varepsilon \ , \tag{5c}$$

$$u_t = \psi(x_t) \quad \forall\, t = 0, ..., T-1 \ . \tag{5d}$$

The equality constraints (5b) are understood to be substituted recursively to eliminate all state variables $x_0, x_1, ..., x_{T-1}$ from the problem. Thus only the state feedback law $\psi(\cdot)$ remains as a free variable in (5).

Remark 3 (Alternative Formulations) *(a) Instead of the sum of expected values, the cost function (5a) can also be defined as a desired quantile of the sum of discounted stage costs. Then the problem formulation corresponds to a minimization of the 'value-at-risk', see e.g. [31]. (b) Multiple chance constraints on the state $\mathbb{X}_p$, each with an individual probability level ε_p, can be included without additional complications. A single chance constraint is considered here for notational simplicity.*

Many practical control problems can be cast in the general form of (5). For example in building climate control [23], the energy consumption of a building should be minimized, while its internal climate is subject to uncertain weather conditions and the occupancy of the building. The comfort range for the room temperatures may occasionally be violated without major harm to the system. Another example is wind turbine control [12], where the power efficiency of a wind turbine should be maximized, while its dynamics are subject to uncertain wind conditions. High stress levels in the blades must not occur too often, in order to maintain a desired fatigue life of the turbine.

3 Scenario-Based Model Predictive Control

The OCP is generally intractable, as it involves an infinite-dimensional decision variable $\psi(\cdot)$ and a large number of constraints (growing with T). Therefore it is common to approximate it by various approaches, such as *Model Predictive Control (MPC)*.

3.1 Stochastic Model Predictive Control (SMPC)

The basic concept of MPC is to solve a tractable counterpart of (5) over a small horizon $N \ll T$ repeatedly at each time step. Only the first input of this solution is applied to the system (1). In Stochastic MPC (SMPC), a *Finite Horizon Optimal Control Problem (FHOCP)* is formulated by introducing chance constraints on the state:

$$\min_{u_{0|t},...,u_{N-1|t}} \ \sum_{t=0}^{N-1} \mathbf{E}\big[\ell\big(x_{i|t}, u_{i|t}\big)\big] \ , \tag{6a}$$

$$\text{s.t.} \ \ x_{i+1|t} = A(\delta_{t+i})x_{i|t} + B(\delta_{t+i})u_{i|t} + w(\delta_{t+i}) \ , \quad x_{0|t} = x_t \quad \forall\, i = 0, ..., N-1 \ , \tag{6b}$$

$$\mathbf{P}\big[x_{i+1|t} \notin \mathbb{X}\big] \leq \varepsilon_i \quad \forall\, i=0,...,N-1 \ , \tag{6c}$$
$$u_{i|t} \in \mathbb{U} \quad \forall\, i=0,...,N-1 \ . \tag{6d}$$

Here $x_{i|t}$ and $u_{i|t}$ denote predictions and plans of the state and input variables made at time t, for i steps into the future. The current measured state x_t is introduced as an initial condition for the dynamics. The predicted states $x_{1|t},...,x_{N|t}$ are understood to be eliminated by recursive substitution of (6b). Note that the predicted states are random by the influence of the uncertainties $\delta_t,...,\delta_{t+N-1}$.

The *probability levels* ε_i in the *chance constraints* (6c) usually coincide with ε from the OCP [14, 23, 30], but they may generally differ [34]. Some formulations also involve chance constraints over the entire horizon [12, 19], or as a combination with robust constraints [10, 18]. Other alternatives of SMPC consider integrated chance constraints [13], or constraints on the expectation of the state [25].

Remark 4 (Terminal Cost) *An optional (convex) terminal cost* $\ell_f : \mathbb{R}^n \to \mathbb{R}_{0+}$ *can be included in the FHOCP [20, 27]. In this case the term*

$$\mathbf{E}\big[\ell_f\big(x_{N|t}\big)\big]$$

will be added to the cost function (6a).

The state feedback law provided by SMPC is given by a receding horizon policy: the current state x_t is substituted into (6b), then the FHOCP is solved for an optimal input sequence $\{u^\star_{0|t},...,u^\star_{N-1|t}\}$, and the current input is set to $u_t := u^\star_{0|t}$. This means that the FHOCP must be solved online at each time step t, using the current measurement of the state x_t.

However, the FHOCP is a stochastic program that remains difficult to solve, except for very special cases. In particular, the feasible set described by chance constraints is generally non-convex, despite of the convexity of $\mathbb{X}$, and hard to determine explicitly. For this reason, a further approximation shall be made by scenario-based optimization.

3.2 Scenario-Based Model Predictive Control (SCMPC)

The basic idea of Scenario-Based MPC (SCMPC) is to compute an optimal finite-horizon input trajectory $\{u'_{0|t},...,u'_{N-1|t}\}$ that is feasible under K of sampled 'scenarios' of the uncertainty. Clearly, the scenario number K has to be selected carefully in order to attain the desired properties of the controller. In this section, the basic setup of SCMPC is discussed, while the selection of a value for K is deferred until Section 4.

More concretely, let $\delta^{(1)}_{i|t},...,\delta^{(K)}_{i|t}$ be i.i.d. samples of δ_{t+i}, drawn at time $t \in \mathbb{N}$ for the prediction steps $i = 0,...,N-1$. For notational convenience, they are combined into *full-horizon samples* $\omega^{(k)}_t := \{\delta^{(k)}_{0|t},...,\delta^{(k)}_{N-1|t}\}$, also called *scenarios*. The *Finite-Horizon Scenario Program (FHSCP)* then reads as follows:

$$\min_{u_{0|t},...,u_{N-1|t}} \sum_{k=1}^{K}\sum_{i=0}^{N-1} \ell\big(x^{(k)}_{i|t}, u_{i|t}\big) \ , \tag{7a}$$

$$\text{s.t. } x^{(k)}_{i+1|t} = A(\delta^{(k)}_{i|t})x^{(k)}_{i|t} + B(\delta^{(k)}_{i|t})u_{i|t} + w(\delta^{(k)}_{i|t}) \ ,$$
$$x^{(k)}_{0|t} = x_t \quad \forall\, i=0,...,N-1,\ k=1,...,K, \tag{7b}$$
$$x^{(k)}_{i+1|t} \in \mathbb{X} \ \ \forall\, i=0,...,N-1,\ k=1,...,K, \tag{7c}$$
$$u_{i|t} \in \mathbb{U} \ \ \forall\, i=0,...,N-1 \ . \tag{7d}$$

The dynamics (7b) provide K different state trajectories over the prediction horizon, each corresponding to one sequence of affine transition maps defined by a particular scenario $\omega^{(k)}_t$. Note that these K state trajectories are not fixed, as they are still subject to the inputs $u_{0|t},...,u_{N-1|t}$. The cost function (7a) approximates (6a) as an average over all K scenarios. The state constraints (7c) are required to hold for K sampled state trajectories over the prediction horizon.

Applying a receding horizon policy, the SCMPC feedback law is defined as follows (see also Figure 1, for $R=0$). At each time step t, the current state measurement x_t is substituted into (7b) and the current input $u_t := u'_{0|t}$ is set to the first of the optimal FHSCP solution $\{u'_{0|t},...,u'_{N-1|t}\}$, called the *scenario solution*.

Unlike many theoretical MPC schemes, SCMPC does not have an inherent guarantee of *recursive feasibility*, in the sense of [22, Sec. 4]. Hence for a proper analysis of the closed-loop system, the following is assumed.

Assumption 5 (Resolvability) *Under the SCMPC regime, each FHSCP admits a feasible solution at every time step* $t \in \mathbb{N}$ *almost surely.*

While Assumption 5 appears to be restrictive from a theoretical point of view, it is often reasonable from a practical point of view. For some applications, such as buildings [23], recursive feasibility may hold intuitively. For other cases, it may be ensured by replacing the state constraints with *soft constraints* [26, Sec. 2]. The assumption here is that MPC remains a useful tool in practice, even for difficult stochastic systems (1) without the possibility of an explicit guarantee of recursive feasibility.

The following are possible alternatives and also convex formulations of (7). The reasoning in each case is based on the theory in [29] and omitted for brevity.

Remark 6 (Alternative Formulations) *(a) Instead of the average cost in (7a), the minimization may concern the cost of a nominal trajectory, as e.g. in [24, 28]; or the average may be taken over any sample size other than K. (b) The inclusion of additional chance constraints into* (7)*, as mentioned in Remark 3(b), is straightforward. The number of scenarios K_p may generally differ between multiple chance constraints p. (c) In case of a value-at-risk formulation, as in Remark 3(a), the average cost in (7a) is replaced by the maximum:*

$$\text{“}\sum_{k=1}^{K}\text{”} \qquad \longrightarrow \qquad \text{“}\max_{k=1,...,K}\text{”}\ ,$$

where the sample size K must be selected according to the desired risk level.

Remark 7 (Control Parameterization) *In the FHSCP, the predicted control inputs $u_{0|t},...,u_{N-1|t}$ may also be parameterized as a weighted sum of basis functions of the uncertainty, as proposed in [32, 33]. In particular, for each time step $i = 1,...,N$, let $q^{(j)}_{i|t} : \Delta^{i-1} \to \mathbb{R}^m$ be a finite set $j \in \{1,...,J_i\}$ of pre-selected basis functions. The terms*

$$u_{0|t} := c_{0|t}\ ,$$
$$u_{i|t} := c_{i|t} + \sum_{j=1}^{J_i} \phi^{(j)}_i q^{(j)}_{i|t}\big(\delta^{(k)}_{0|t},...,\delta^{(k)}_{i-1|t}\big) \quad \forall\, i = 1,...,N-1,$$

can be substituted into problem (7)*. The corrective control inputs $c_{0|t},...,c_{N-1|t} \in \mathbb{R}^m$ become the new decision variables, and the weights $\phi^{(j)}_i \in \mathbb{R}$ for $i = 0,...,N-1$ and $j = 1,...,J_i$ can be determined on-line or off-line.*

A control parameterization with an increasing number of basis functions $J_1,...,J_{N-1}$ generally improves the quality of the SCMPC feedback. At the same time, it increases the number of decision variables and hence the computational complexity; see [32, 33] for more details.

Given the sampled scenarios, (7) is a convex optimization program for which efficient solution algorithms exist, depending on its structure [5]. In particular, if $\mathbb{X}$ and $\mathbb{U}$ are polytopic (respectively ellipsoidal) sets, then the FHSCP has linear (second-order cone) constraints. If the stage cost is either (4a,b), then the FHSCP has a reformulation with a linear objective function, using auxiliary variables. If the stage cost is (4c), then the FHSCP can be expressed as a quadratic program. More details on these formulation procedures are found in [20, pp. 154 f.].

3.3 A-Posteriori Scenario Removal

One merit of SCMPC is that it renders the uncertain control system (6b) into multiple deterministic affine systems (7b) by substituting particular scenarios. This significantly simplifies the solution to the FHSCP, as compared to the FHOCP. However, these scenarios lead to a *randomization* of the SCMPC feedback law. In particular, the closed-loop system may occasionally show an erratic behavior due to highly unlikely outliers in the sampled scenarios.

This randomization effect can be mitigated by an a-posteriori scenario removal; see [9, Sec. 1] for the motivation and further details. A-posteriori scenario removal allows for $R > 0$ sampled *state constraints* (7c) to be removed *after* the outcomes of all samples have been observed.

A key point to notice is the fundamental difference between decreasing K by 1 and setting R to 1: the first case removes an *unobserved* sample and the second case an *observed* sample from the FHSCP. While both cases have a relaxing effect on the constraints of the FHSCP, the effect of the second case is generally higher. Therefore when increasing R, the sample size K must also be increased (appropriately, generally more than R), in order to maintain the same level of constraint violations. In other words, for a specified violation level $\varepsilon \in (0, 1)$, R has to be varied in proper combination with K.

Remark 8 (Admissible Sample-Removal Pair) *An* admissible sample-removal pair *is considered to be a combination (K, R) that does not exceed the desired constraint violation level $\varepsilon \in (0, 1)$. The exact choice of admissible sample-removal pairs (K, R) is the subject of Section 4.*

For now, suppose that an admissible sample removal pair (K, R) is given. After K scenarios have been sampled, the selection of the R removed scenarios is performed by a *(scenario) removal algorithm* [9, Def. 2.1].

Definition 9 (Removal Algorithm) *(a) For each $\xi \in \mathbb{R}^n$, the* (scenario) removal algorithm *$\mathcal{A}_\xi : \Delta^{NK} \to \Delta^{N(K-R)}$ is a deterministic function selecting $(K-R)$ out of K scenarios $\Omega_t := \{\omega^{(1)}_t,...,\omega^{(K)}_t\}$. (b) The selected scenarios at time step t shall be denoted by*

$$\Omega'_t := \mathcal{A}_{x_t}\big(\omega^{(1)}_t,...,\omega^{(K)}_t\big)\ .$$

Definition 9 is very general, in the sense that it covers a great variety of possible scenario removal algorithms. However, the most common and practical algorithms are described below:

Optimal Removal: The FHSCP is solved for all possible combinations of choosing R out of K scenarios. Then the combination that yields the lowest cost function value of all the solutions is selected. This requires the solution to K choose R instances of the FHSCP, a complexity that is usually prohibitive for larger values of R.

Greedy Removal: The FHSCP is first solved with all K scenarios. Then, in each of R consecutive steps, the state constraints of a single scenario are removed which yield the biggest improvement, either in the total cost or in the first stage cost. Thus the procedure terminates after solving $KR - R(R-1)/2$ instances of FHSCP.

Marginal Removal: The FHSCP is first solved with the state constraints of all K scenarios. Then, in each of R consecutive steps, the state constraints of a single scenario are removed based on the highest Lagrange multiplier. Hence the procedure requires the solution to K instances of FHSCP.

Figure 1 depicts an algorithmic overview of SCMPC, for the general case with scenario removal $R > 0$. For the case without scenario removal, consider $R = 0$ and hence $\Omega'_t := \{\omega_t^{(1)}, ..., \omega_t^{(K)}\}$.

At every time step t, perform the following steps:

1. Measure current state x_t.
2. Extract K scenarios $\Omega_t = \{\omega_t^{(1)}, ..., \omega_t^{(K)}\}$.
3. Remove R scenarios via $\mathcal{A}_{x_t}$, and then solve FHSCP with only the state constraints of the remaining scenarios in Ω'_t.
4. Apply the first input of the scenario solution $u_t := u'_{0|t}$ to the system.

Fig. 1. Schematic overview of the SCMPC algorithm, for the case with scenario removal ($R > 0$) and without scenario removal ($R = 0$).

4 Problem Structure and Sample Complexity

For the SCMPC algorithm described in Section 3, the values of K and R remain to be specified. This section contains the theoretical foundations for the proper choice of an admissible sample-removal pair (K, R). Their values generally depend on the control system and the constraints. K is referred to as the *sample complexity* of the SCMPC problem.

For some intuition about this problem, suppose that $R \geq 0$ is fixed and the sample size K is increased. This means that the solution to the FHSCP becomes robust to more scenarios, with the following consequences. First, the average-in-time state constraint violations (3) decrease, in general. Therefore the state constraint will translate into a lower bound on K. Second, the computational complexity increases as well as the average-in-time closed-loop cost (2), in general. Therefore the objective is to choose K as small as possible, and ideally equal to its lower bound.

The higher the number of removed constraints $R \geq 0$, the higher will be the lower bound on K, in order for the state constraints (3) to be satisfied. Now consider pairs (R, K) of removed constraints R together with their corresponding lower bounds K, which equally satisfy the state constraints (3). For the intuition, suppose R is increased, so K increases as well. Then the computational complexity grows, due to a higher number of constraints in the FHSCP and due to the removal algorithm. At the same time, the solution to the FHSCP improves, in the sense that it converges asymptotically to the solution of the FHOCP with increasing R [9, Sec. 6]. Therefore the average-in-time closed-loop cost (2) can be expected to decrease. If the computational resources are fixed, R can thus be chosen as the highest value permitting the execution of the SCMPC algorithm (Figure 1) within the allotted time frame.

4.1 Support Rank

According to the classic scenario approach [8,9], the relevant quantity for determining the sample size K for a single chance constraint (with a fixed R) is the number of *support constraints* [8, Def. 2.1]. In fact, K grows with the (unknown) number of support constraints, so the goal is to obtain a tight upper bound. For the classic scenario approach, this upper bound is given by the dimension of the decision space [8, Prop. 2.2], i.e. Nm in the case of the FHSCP.

The FHSCP is a multi-stage stochastic program, with multiple chance constraints (namely N, one per stage). This requires an extension to the classic scenario approach; the reader is referred to [29] for more details. Now each chance constraint contributes an individual number of support constraints, to which an upper bound must be obtained. These individual upper bounds are provided by the *support rank* of each chance constraint [29, Def. 3.6].

Definition 10 (Support Rank) *(a) The* unconstrained subspace $\mathcal{L}_i$ *of a constraint* $i \in \{0, ..., N-1\}$ *in (7c) is the largest (in the set inclusion sense) linear subspace of the search space* $\mathbb{R}^{Nm}$ *that remains unconstrained by all sampled instances of* i*, almost surely. (b) The* support rank *of a constraint* $i \in \{0, ..., N-1\}$ *in (7c) is*

$$\rho_i := Nm - \dim \mathcal{L}_i \ ,$$

where $\dim \mathcal{L}_i$ *represents the dimension of the unconstrained subspace* $\mathcal{L}_i$*.*

Note that the support rank is an inherent property of a particular chance constraint and it is not affected by the simultaneous presence of other constraints. Hence the set of constraints of the FHSCP may change, for instance, due to the reformulations of Remark 3.

Besides the extension to multiple chance constraints, the support rank has the merit of significantly reducing the upper bound on the number of support constraints. Indeed, the following two lemmas replace the classic upper bound Nm with much lower values, such as $l \leq n$ or m, depending on the problem structure.

For systems affected by *additive* disturbances only, the support rank of any state constraint in the FHSCP is given by the support rank $l \leq n$ of $\mathbb{X}$ in $\mathbb{R}^n$ (i.e., the co-dimension of the largest linear subspace that is unconstrained by $\mathbb{X}$).

Lemma 11 (Pure Additive Disturbances) *Let $l \leq n$ be the support rank of $\mathbb{X}$ and suppose that $A\big(\delta_{i|t}^{(k)}\big) \equiv A$ and $B\big(\delta_{i|t}^{(k)}\big) \equiv B$ are constant and the control is not parameterized (as in Remark 7). Then the support rank of any state constraint $i \in \{0, ..., N-1\}$ in (7c) is at most l.*

For systems affected by additive *and* multiplicative disturbances, Lemma 11 no longer holds. However, it will be seen that for the desired closed-loop properties, the relevant quantity for selecting the sample size K is the support rank ρ_1 of the state constraint on $x_{1|t}$ only. For this first predicted step, the support rank is restricted to at most m, under both additive and multiplicative disturbances.

Lemma 12 (Additive and Multiplicative Disturbances) *The support rank ρ_1 of constraint $i = 1$ in (7c) is at most m.*

For the sake of readability, the proofs of Lemmas 11 and 12 can be found in Appendix A.

Note that these results effectively decouple the support rank, and hence the sample size K, from the horizon length N. The result of Lemma 12 holds also for the parameterized control laws of Remark 7. In this case, it decouples the sample size K from the number of basis functions J_i for all stages $i = 1, ..., N-1$.

Tighter bounds of ρ_1 than those in Lemmas 11 and 12 may exist, resulting from a special structure of the system (1) and/or the state constraint set $\mathbb{X}$. The basic insights to exploit this can be found in the Appendix A and [29].

4.2 Sample Complexity

This section describes the selection of the admissible sample-removal pair (K, R), based on a bound of the support rank ρ_1. Throughout this section, the initial state x_t is considered to be fixed to an arbitrary value.

Let $V_t|x_t$ denote the *(first step) violation probability*, i.e. the probability with which the first predicted state falls outside of $\mathbb{X}$:

$$V_t|x_t := \mathbf{P}\big[A(\delta_t)x_t + B(\delta_t)u'_{0|t} + w(\delta_t) \notin \mathbb{X} \,\big|\, x_t\big] \ . \quad (8)$$

Recall that $u'_{0|t}$ denotes the first input of the scenario solution $\{u'_{0|t}, ..., u'_{N-1|t}\}$. Clearly, $u'_{0|t}$ and $V_t|x_t$ depend on the particular scenarios $\Omega_t = \{\omega_t^{(1)}, ..., \omega_t^{(K)}\}$ extracted by the SCMPC algorithm at time t (see Figure 1). The notation $u'_{0|t}(\Omega_t)$ and $V_t|x_t(\Omega_t)$ shall be used occasionally to emphasize this fact.

The violation probability $V_t|x_t(\Omega_t)$ can be considered as a random variable on the probability space $(\Delta^{KN}, \mathbf{P}^{KN})$, with support in $[0, 1]$. Here Δ^{KN} and $\mathbf{P}^{KN}$ denote the KN-th product of the set Δ and the measure $\mathbf{P}$, respectively. For distinction, the expectation operator on $(\Delta, \mathbf{P})$ is denoted $\mathbf{E}$, and that on $(\Delta^{KN}, \mathbf{P}^{KN})$ is denoted $\mathbf{E}^{KN}$.

The distribution of $V_t|x_t(\Omega_t)$ is unknown, being a complicated function of the entire control problem (6) and the removal algorithm $\mathcal{A}_{x_t}$. However, it is possible to derive the following upper bound on this distribution.

Lemma 13 (Upper Bound on Distribution) *Let Assumptions 1, 2, 5 hold and $x_t \in \mathbb{R}^n$ be an arbitrary initial state. For any violation level $\nu \in [0, 1]$,*

$$\mathbf{P}^{KN}\big[V_t|x_t(\Omega_t) > \nu\big] \leq U_{K,R,\rho_1}(\nu)\,, \quad (9a)$$

$$U_{K,R,\rho_1}(\nu) := \min\Big\{1, \binom{R+\rho_1-1}{R} \mathrm{B}\big(\nu; K, R+\rho_1-1\big)\Big\}\,, \quad (9b)$$

where $\mathrm{B}(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ represents the beta distribution function [1, frm. 26.5.3, 26.5.7],

$$\mathrm{B}\big(\nu; K, R+\rho_1-1\big) := \sum_{j=0}^{R+\rho_1-1} \binom{K}{j} \nu^j (1-\nu)^{K-j} \,.$$

Proof. The proof is a straightforward extension of [29, Thm. 6.7], where the bound on $V_t|x_t(\Omega_t)$ is saturated at 1. ■

This paper exploits the result of Lemma 13 to obtain an upper bound on the expectation

$$\mathbf{E}^{KN}\big[V_t \,\big|\, x_t\big] := \int_{\Delta^{KN}} V_t|x_t(\Omega_t)\, d\,\mathbf{P}^{KN} \quad . \quad (10)$$

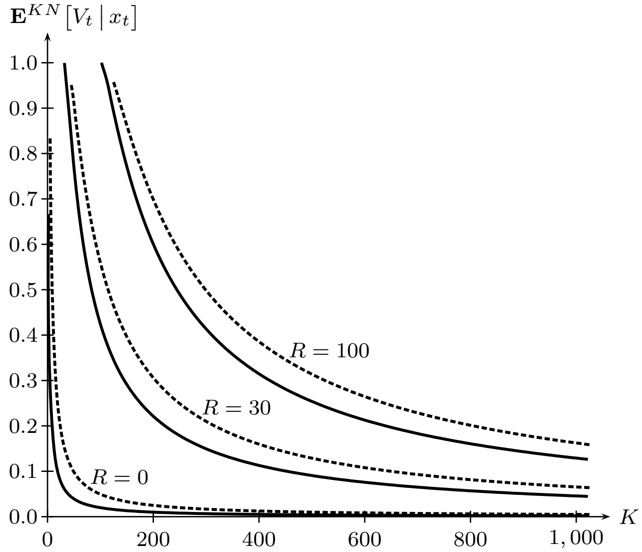


Fig. 2. Upper bound on the expected violation probability $\mathbf{E}^{KN}\big[V_t \,\big|\, x_t\big]$, as a function of the sample size K, for different scenario removals R and support dimensions $\rho_1 = 2$ (solid lines) and $\rho_1 = 5$ (dashed lines).

A reformulation via the indicator function $\mathbf{1} : \Delta^{KN} \to \{0, 1\}$ yields that

$$\begin{aligned}\mathbf{E}^{KN}\big[V_t \,\big|\, x_t\big] &= \int_{[0,1]} \int_{\Delta^{KN}} \mathbf{1}\big(V_t | x_t(\Omega_t) > \nu\big)\, d\,\mathbf{P}^{KN}\, d\nu \\ &= \int_{[0,1]} \mathbf{P}^{KN}\big[V_t | x_t(\Omega_t) > \nu\big]\, d\nu \\ &\leq \int_{[0,1]} U_{K,R,\rho_1}(\nu)\, d\nu \ . \qquad (11)\end{aligned}$$

Definition 14 (Admissible Sample-Removal Pair) *A sample-removal pair* (K, R) *is* admissible *if its substitution into* (11) *yields* $\mathbf{E}^{KN}\big[V_t \,\big|\, x_t\big] \leq \varepsilon$.

Whether a given sample-removal pair (K, R) is admissible can be tested by performing the one-dimensional numerical integration (11). It can easily be seen that the integral value (11) monotonically decreases with K and monotonically increases with R. Hence, if either K or R is fixed, an admissible sample-removal pair (K, R) can be determined, e.g., by a bisection method. Moreover, if R is fixed, there always exist K large enough to generate an admissible pair (K, R).

Remark 15 (No Scenario Removal) *If* $R = 0$, *the integration* (11) *can be replaced by the exact analytic formula*

$$\mathbf{E}^{KN}\big[V_t \,\big|\, x_t\big] \leq \frac{\rho_1}{K+1} \ . \qquad (12)$$

Figure 2 illustrates the monotonic relationship of the upper bound (11) in K and R. Supposing that $R = 0, 30, 100$ is fixed, the corresponding admissible pair (K, R) can be found by moving along the graphs until the desired violation level ε is reached. The solid and the dashed line correspond to the exemplary support dimensions $\rho_1 = 2$ and $\rho_1 = 5$.

4.3 Closed-Loop Properties

This section analyzes the closed-loop properties of the control system under the SCMPC law for an admissible sample-removal pair (K, R). To this end, the underlying stochastic process is first described. Recall that

- $x_0, ..., x_{T-1}$ is the closed-loop trajectory, where x_t depends on all past uncertainties $\delta_0, ..., \delta_{t-1}$ as well as all past scenarios $\Omega_0, ..., \Omega_{t-1}$;
- $V_0, ..., V_{T-1}$ are the violation probabilities, where V_t depends on x_t and Ω_t, and hence on $\Omega_0, ..., \Omega_t$ and $\delta_0, ..., \delta_{t-1}$;
- $M_0, ..., M_{T-1}$ indicate the actual violation of the constraints, where M_t depends on x_{t+1}, and hence on $\Omega_0, ..., \Omega_t$ and $\delta_0, ..., \delta_t$.

At each time step t, there are a total of $D := (KN + 1)$ random variables, namely the scenarios together with the disturbance $\{\delta_t, \Omega_t\} \in \Delta^{(KN+1)} = \Delta^D$. In order to simplify notations, define

$$\mathcal{F}_t := \{\delta_0, \Omega_0, ..., \delta_t, \Omega_t\} \in \Delta^{(t+1)D} \ ,$$

for any $t \in \{0, ..., T-1\}$. These auxiliary variables allow for the random variables $x_t(\mathcal{F}_{t-1})$, $V_t(\mathcal{F}_{t-1}, \Omega_t)$, $M_t(\mathcal{F}_t)$ to be expressed in terms of their elementary uncertainties. Moreover, let $\mathbf{P}^{(t+1)D}$ denote the probability measure and $\mathbf{E}^{(t+1)D}$ the expectation operator on $\Delta^{(t+1)D}$, for any $t \in \{0, ..., T-1\}$.

Observe that $M_t \in \{0, 1\}$ is a Bernoulli random variable with (random) parameter V_t, because

$$\begin{aligned}\mathbf{E}\big[M_t \,\big|\, \mathcal{F}_{t-1}, \Omega_t\big] &= \int_\Delta M_t(\mathcal{F}_t)\, d\,\mathbf{P}(\delta_t) \\ &= V_t(\mathcal{F}_{t-1}, \Omega_t) \qquad (13)\end{aligned}$$

for any values of $\mathcal{F}_{t-1}, \Omega_t$.

Theorem 16 *Let Assumptions 1, 2, 5 hold and* (K, R) *be an admissible sample-removal pair. Then the expected time-average of closed-loop constraint violations* (3) *remains below the specified level* ε,

$$\mathbf{E}^{TD}\Big[\frac{1}{T}\sum_{t=0}^{T-1} M_t\Big] \leq \varepsilon \ , \qquad (14)$$

for any $T \in \mathbb{N}$.

Proof. By linearity of the expectation operator,

$$\begin{aligned}
&\mathbf{E}^{TD}\big[\frac{1}{T}\big(M_0+M_1+...+M_{T-1}\big)\big] \\
&=\frac{1}{T}\big(\mathbf{E}^{D}\big[M_0\big]+\mathbf{E}^{2D}\big[M_1\big]+...+\mathbf{E}^{TD}\big[M_{T-1}\big]\big) \\
&=\frac{1}{T}\big(\mathbf{E}^{D-1}\big[V_0\big]+\mathbf{E}^{2D-1}\big[V_1\big]+...+\mathbf{E}^{TD-1}\big[V_{T-1}\big]\big)\,,
\end{aligned}$$

by virtue of (13). Moreover, for any $t\in\{0,...,T-1\}$,

$$\mathbf{E}^{(t+1)D-1}\big[V_t\big]=\int_{\Delta^{tD}}\underbrace{\mathbf{E}^{D-1}\big[V_t\,\big|\,\mathcal{F}_{t-1}\big]}_{\leq\varepsilon}\,d\,\mathbf{P}^{tD}\leq\varepsilon\,,$$

where the integrand is pointwise upper bounded by ε because (K,R) is an admissible sample-removal pair. ■

Theorem 16 shows that the chance constraints of the OCP can be expected to be satisfied over any finite time horizon T. The next Lemma 17 sets the stage for an even stronger result, Theorem 18, showing that the chance constraint is satisfied almost surely as $T\to\infty$.

Lemma 17 *If Assumptions 1, 2, 5 hold, then*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\Big(M_t-\mathbf{E}^{D-1}\big[V_t\big|\mathcal{F}_{t-1}\big]\Big)=0 \qquad (15)$$

almost surely.

Proof. For any $t\in\mathbb{N}$, define $Z_t:=M_t-\mathbf{E}^{D-1}\big[V_t\big|\mathcal{F}_{t-1}\big]$ and observe that

$$\begin{aligned}
&\mathbf{E}^{D}\big[Z_t\big|\mathcal{F}_{t-1}\big] && (16)\\
&=\mathbf{E}^{D}\big[M_t\big|\mathcal{F}_{t-1}\big]-\mathbf{E}^{D}\big[\mathbf{E}^{D-1}\big[V_t\big|\mathcal{F}_{t-1}\big]\big|\mathcal{F}_{t-1}\big] \\
&=\mathbf{E}^{D}\big[M_t\big|\mathcal{F}_{t-1}\big]-\mathbf{E}^{D-1}\big[V_t\big|\mathcal{F}_{t-1}\big] \\
&=0\ , && (17)
\end{aligned}$$

by virtue of (13). In probabilistic terms, this says that $\{Z_t\}_{t\in\mathbb{N}}$ is a sequence of martingale differences. Moreover,

$$\sum_{t=0}^{\infty}\frac{1}{(t+1)^2}\,\mathbf{E}^{D}\big[Z_t^2\big|\mathcal{F}_{t-1}\big]<\infty \qquad (18)$$

almost surely, because $|Z_t|\leq 1$ is bounded for $t\in\mathbb{N}$. Therefore [16, Thm. 2.17] can be applied, which yields that

$$\sum_{t=0}^{T-1}\frac{1}{t+1}Z_t \qquad (19)$$

converges almost surely as $T\to\infty$. The result (15) now follows by use of Kronecker's Lemma, [16, p. 31]. ■

Note that Lemma 17 does not imply that

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}M_t=\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbf{E}^{D-1}\big[V_t\big|\mathcal{F}_{t-1}\big] \qquad (20)$$

almost surely, because it is not clear that the right-hand side converges almost surely. However, if it converges almost surely, then (20) holds.

Theorem 18 *Let Assumptions 1, 2, 5 hold and (K,R) be an admissible sample-removal pair. Then*

$$\limsup_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}M_t\leq\varepsilon \qquad (21)$$

almost surely.

Proof. From Lemma 17,

$$\begin{aligned}
0&=\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\Big(M_t-\mathbf{E}^{D-1}\big[V_t\big|\mathcal{F}_{t-1}\big]\Big) \\
&\geq\limsup_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\big(M_t-\varepsilon\big) \\
&=\limsup_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}M_t-\varepsilon && (22)
\end{aligned}$$

almost surely, where the second line follows from Definition 14. ■

5 Numerical Example

5.1 System Data

Consider the stochastic linear system

$$x_{t+1}=\begin{bmatrix}0.7 & -0.1(2+\theta_t)\\ -0.1(3+2\theta_t) & 0.9\end{bmatrix}x_t+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}u_t+\begin{bmatrix}w_t^{(1)}\\ w_t^{(2)}\end{bmatrix},$$

where $x_0=[1\ 1]^{\mathrm{T}}$. Here $\theta_t\sim\mathcal{U}\big([0,1]\big)$ is uniformly distributed on the interval $[0,1]$ and $w_t^{(1)},w_t^{(2)}\sim\mathcal{N}(0,0.1)$ are normally distributed with mean 0 and variance 0.1. The inputs are constrained as

$$\mathbb{U}:=\big\{\upsilon\in\mathbb{R}^2\,\big|\,|\upsilon^{(1)}|\leq 5\,\wedge\,|\upsilon^{(2)}|\leq 5\big\}\ .$$

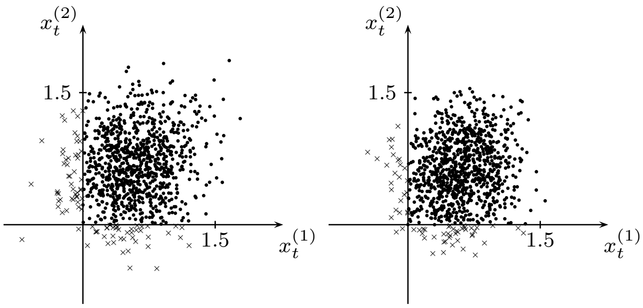
Moreover, two state constraints are considered,

$$\mathbb{X}_1:=\big\{\xi\in\mathbb{R}^2\,\big|\,\xi^{(1)}\geq 1\big\}\ ,\ \ \mathbb{X}_2:=\big\{\xi\in\mathbb{R}^2\,\big|\,\xi^{(2)}\geq 1\big\}\ ,$$

either individually or jointly as $\mathbb{X}:=\mathbb{X}_1\cap\mathbb{X}_2$. The stage cost function is chosen to be of the quadratic form (4c), with the weights $Q_\ell:=I$ and $R_\ell:=I$. The MPC horizon is set to $N:=5$.

5.2 *Joint Chance Constraint*

The support rank of the joint chance constraint $\mathbb{X}$ is easily obtained as $\rho_1 = 2$. Figure 3 depicts a phase plot of the closed-loop system trajectory, for two admissible sample-removal pairs (a) $(19, 0)$ and (b) $(1295, 100)$, corresponding to $\varepsilon = 10\%$. Instances in which the state trajectory leaves $\mathbb{X}$ are indicated with a cross. Note that the distributions are centered around a similar mean in both cases, however the case $R = 0$ features stronger outliers than $R = 100$.



(a) Case $R = 0$ (b) Case $R = 100$

Fig. 3. Phase plot of closed-loop system trajectory (cross: violating states; dots: other states). The axis lines mark the boundary of the feasible set $\mathbb{X}$.

Table 1 shows the empirical results of a simulation of the closed-loop system over $T = 10,000$ time steps. Note that there is essentially no difference between the specified and the empirical violation levels, ε and V_{avg}, in the case of no removals ($R = 0$). A minor difference is present for small removal sizes, disappearing asymptotically as $R \to \infty$. At the same time, the reduction of the average closed-loop cost ℓ_{avg} is minor for this example, while the standard deviation ℓ_{std} is affected significantly.

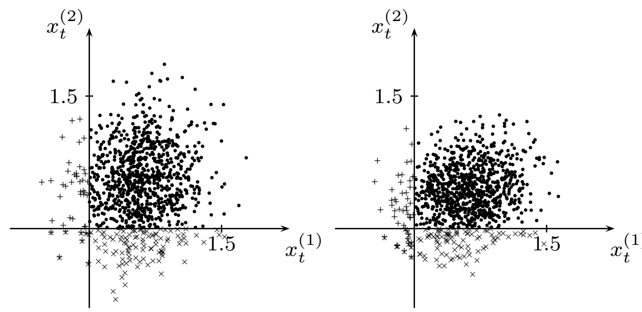
$\varepsilon = 10\%$	$R = 0$	$R = 50$	$R = 100$	$R = 500$
K	19	702	1,295	5,723
V_{avg}	9.87%	7.37%	8.06%	8.74%
ℓ_{avg}	3.78	3.75	3.72	3.68
ℓ_{std}	0.54	0.44	0.42	0.37

Table 1
Joint chance constraint: closed-loop results for mean violation level V_{avg}, mean stage cost ℓ_{avg}, and standard deviation of stage costs ℓ_{std}.

To highlight the impact of the method presented in this paper, the results of Table 1 can be compared to those of previous SCMPC approaches [6, 28]. The sample size is 19 (compared to about 400), and the empirical share of constraint violations in closed-loop is 9.87% (compared to about 0.5%). These numbers become even worse when longer horizons are considered; e.g. for $N = 20$, previous approaches require about 900 samples and yield about 0.2% violations.

5.3 *Individual Chance Constraints*

For the same example, the two chance constraints $\mathbb{X}_1$ and $\mathbb{X}_2$ are now considered separately, with the individual probability levels $\varepsilon_1 = 5\%$ and $\varepsilon_2 = 10\%$. Each support rank is bounded by $\rho_1 = 1$. Figure 4 depicts a phase plot of the closed-loop system trajectory, for the admissible sample-removal pairs (a) $(19, 0)$, $(9, 0)$ and (b) $(2020, 100)$, $(1010, 100)$.



(a) Case $R_1 = R_2 = 0$ (b) Case $R_1 = R_2 = 100$

Fig. 4. Phase plot of closed-loop system trajectory (plus, cross, star: violating states of $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_1$ and $\mathbb{X}_2$; dots: other states). The axis lines mark the boundaries of the feasible sets $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively.

Table 2 shows the empirical results of a simulation of the closed-loop system over $T = 10,000$ time steps. Note that the empirical violation levels $V_{\text{avg},1}$, $V_{\text{avg},2}$ match very well to the specifications $\varepsilon_1, \varepsilon_2$ in all cases. As in the previous example, the reduction of the average closed-loop cost ℓ_{avg} is minor, while the standard deviation ℓ_{std} is affected significantly.

$\varepsilon_1 = 5\%$, $\varepsilon_2 = 10\%$	$R_1 = R_2 = 0$	$R_1 = R_2 = 50$	$R_1 = R_2 = 100$
K_1	19	1,020	2,020
K_2	9	510	1,010
$V_{\text{avg},1}$	5.14%	4.84%	4.95%
$V_{\text{avg},2}$	9.94%	9.81%	9.93%
ℓ_{avg}	3.67	3.62	3.51
ℓ_{std}	0.54	0.46	0.42

Table 2
Individual chance constraints: closed-loop results for mean violation levels $V_{\text{avg},1}$ and $V_{\text{avg},2}$ of $\mathbb{X}_1$ and $\mathbb{X}_2$, mean stage cost ℓ_{avg}, and standard deviation of stage costs ℓ_{std}.

6 Conclusion

The paper has presented new results on Scenario-Based Model Predictive Control (SCMPC). By focusing on the average-in-time probability of constraint violations and by exploiting the multi-stage structure of the finite-horizon optimal control problem (FHOCP), the number of scenarios has been greatly reduced compared to previous approaches. Moreover, the possibility to adopt a-posteriori constraint removal strategies has also been accommodated. Due to its computational efficiency, the presented approach paves the way for a tractable application of Stochastic Model Predictive Control (SMPC) to large-scale problems with hundreds of decision variables.

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A Proof of Lemmas 11 and 12

The particular bounding arguments follow rather easily after some general observations on the support rank. Pick any state constraint $i \in \{1, ..., N\}$ from (7c). Recursively substituting the dynamics (7b), the constrained state can be expressed as

$$x_{i|t}^{(k)} = \big(A_{i|t}^{(k)} \cdot ... \cdot A_{0|t}^{(k)}\big) x_t + \\ + \bar{A}_{i|t}^{(k)} \bar{B}_{i|t}^{(k)} \begin{bmatrix} u_{0|t} \\ \vdots \\ u_{N-1|t} \end{bmatrix} + \bar{A}_{i|t}^{(k)} \begin{bmatrix} w_{0|t}^{(k)} \\ \vdots \\ w_{i-1|t}^{(k)} \end{bmatrix} , \quad \text{(A.1a)}$$

$$\bar{A}_{i|t}^{(k)} := \begin{bmatrix} A_{i|t}^{(k)} \cdot ... \cdot A_{1|t}^{(k)} \\ \vdots \\ A_{1|t}^{(k)} \\ \mathrm{I} \end{bmatrix}^{\mathrm{T}} , \quad \text{(A.1b)}$$

$$\bar{B}_{i|t}^{(k)} := \begin{bmatrix} B_{0|t}^{(k)} & 0 & \dots & 0 & 0 \dots 0 \\ 0 & B_{1|t}^{(k)} & \dots & 0 & 0 \dots 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \dots 0 \\ 0 & 0 & \dots & B_{i|t}^{(k)} & 0 \dots 0 \end{bmatrix} , \quad \text{(A.1c)}$$

where $I \in \mathbb{R}^{n \times n}$ denotes the identity matrix, and for any $i = 0, ..., N-1$ the following abbreviations are used:

$$A_{i|t}^{(k)} := A\big(\delta_{i|t}^{(k)}\big), \;\; B_{i|t}^{(k)} := B\big(\delta_{i|t}^{(k)}\big), \;\; w_{i|t}^{(k)} := w\big(\delta_{i|t}^{(k)}\big) \; .$$

Let $l \leq n$ be the support rank of $\mathbb{X}$, i.e. the co-dimension of the largest linear subspace that is unconstrained by $\mathbb{X}$. Then there exists a projection matrix $P \in \mathbb{R}^{l \times n}$ such that for each $x \in \mathbb{R}^n$

$$x \in \mathbb{X} \quad \Longleftrightarrow \quad Px \in P\mathbb{X} := \big\{ P\xi \,\big|\, \xi \in \mathbb{X} \big\} \; .$$

For example, if the state constraint concerns only the first two elements of the state vector, then $l = 2$ and $P \in \mathbb{R}^{2 \times n}$ may contain the first two unit vectors $e_1, e_2 \in \mathbb{R}^n$ as its rows.

Proof of Lemma 11

If $A\big(\delta_{i|t}^{(k)}\big) \equiv A$ and $B\big(\delta_{i|t}^{(k)}\big) \equiv B$ are constant for all $i \in \{0, ..., N-1\}$, then (A.1a) reduces to

$$\underbrace{\big[PA^{i-1}B \; \dots \; P \;\; 0 \; \dots\big]}_{\mathrm{rank}(\cdot) \leq l} \begin{bmatrix} u_{0|t} \\ \vdots \\ u_{N-1|t} \end{bmatrix} + PA^i x_t + \\ + \big[PA^{i-1}B \; \dots \; P\big] \begin{bmatrix} w_{0|t}^{(k)} \\ \vdots \\ w_{i-1|t}^{(k)} \end{bmatrix} \in P\mathbb{X} \; , \quad \text{(A.2)}$$

for any $i \in \{1, ..., N\}$. The rank of the first matrix of dimension $l \times Nm$ can be at most l, and therefore it has a null space of dimension at least $Nm - l$. The disturbance has no effect on this null space, because it enters only through the third, additive term in (A.2). Hence this null space is clearly an unconstrained subspace of the constraint and $\rho_i \leq l \leq n$ for all $i \in \{1, ..., N\}$, proving Lemma 11.

Proof of Lemma 12

Consider the first state constraint $i = 1$ of (7c). Here (A.1a) reduces to

$$\underbrace{\big[P\bar{B}_{0|t}^{(k)} \;\; 0 \; \dots \; 0\big]}_{\mathrm{rank}(\cdot) \leq m} \begin{bmatrix} u_{0|t} \\ \vdots \\ u_{N-1|t} \end{bmatrix} + PA_{0|t}^{(k)} x_t + \\ + Pw_{0|t}^{(k)} \in P\mathbb{X} \; . \quad \text{(A.3)}$$

The rank of the first matrix can here be at most m for all outcomes of $\bar{B}_{0|t}^{(k)}$, because the last $(N-1)m$ variables in the decision vector are always in its null space. Hence $\rho_1 \leq m$ in all cases, proving Lemma 12.

Parameterized Control Laws

For the case of parameterized control laws as in Remark 7, it will be shown that the argument of Lemma 12 continues to apply. Define for any $i = 1, ..., N-1$

$$Q_{i|t}^{(k)} := \underbrace{\begin{bmatrix} q_{i|t}^{(1)} & q_{i|t}^{(2)} & \dots & q_{i|t}^{(J_i)} \end{bmatrix}}_{\in \mathbb{R}^{m \times J_i}} , \qquad \Phi_{i|t} := \underbrace{\begin{bmatrix} \phi_{i|t}^{(1)} \\ \vdots \\ \phi_{i|t}^{(J_i)} \end{bmatrix}}_{\in \mathbb{R}^{J_i}} ,$$

where $q_{i|t}^{(j)} := q_{i|t}^{(j)}\big(\delta_{0|t}^{(k)}, ..., \delta_{i|t}^{(k)}\big)$ is used as an abbreviation. Then the vector of control inputs under scenario $k = 1, ..., K$ can be put into the affine expression

$$\begin{bmatrix} u_{0|t} \\ u_{1|t}^{(k)} \\ \vdots \\ u_{N-1|t}^{(k)} \end{bmatrix} = \begin{bmatrix} c_{0|t} \\ c_{1|t} \\ \vdots \\ c_{N-1|t} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & Q_{1|t}^{(k)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{N-1|t}^{(k)} \end{bmatrix}}_{=: \bar{Q}_t^{(k)}} \underbrace{\begin{bmatrix} \Phi_{0|t} \\ \Phi_{1|t} \\ \vdots \\ \Phi_{N-1|t} \end{bmatrix}}_{=: \bar{\Phi}_t} .$$

Substitute this for the original decision vector into (A.3). In the case of off-line optimization, where $\bar{\Phi}_t$ is fixed, and in the case of on-line optimization, where $\bar{\Phi}_t$ is part of the decision variables, the same rank argument from before applies.

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